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This and the next issue are dedicated to the memory of two distinguished members of Editorial Board

Academician Fikret Vajzović, member of ANUBiH and Professor Harry I. Miller, foreign member of ANUBiH, who was the Editor in Chief, for many years.
GLOBAL DYNAMICS OF CERTAIN MIX MONOTONE DIFFERENCE EQUATION VIA CENTER MANIFOLD THEORY AND THEORY OF MONOTONE MAPS

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Dedicated to the memory of Accademiocrats Harry I. Miller and Fikret Vajzović, our teachers and supporters.

ABSTRACT. We investigate the global dynamics of the following rational difference equation of second order

\[ x_{n+1} = \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + f}, \quad n = 0, 1, \ldots, \]

where the parameters \( A \) and \( E \) are positive real numbers and the initial conditions \( x_{-1} \) and \( x_0 \) are arbitrary non-negative real numbers such that \( x_{-1} + x_0 > 0 \). The transition function associated with the right-hand side of this equation is always increasing in the second variable and can be either increasing or decreasing in the first variable depending on the parametric values. The unique feature of this equation is that the second iterate of the map associated with this transition function changes from strongly competitive to strongly cooperative. Our main tool for studying the global dynamics of this equation is the theory of monotone maps while the local stability is determined by using center manifold theory in the case of the nonhyperbolic equilibrium point.

1. INTRODUCTION

In this paper, we investigate the global dynamics of the following difference equation

\[ x_{n+1} = \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + f}, \quad n = 0, 1, \ldots, \] (1)

where \( A, E, f \in (0, \infty) \) and where the initial conditions \( x_{-1} \) and \( x_0 \) are arbitrary non-negative real numbers such that \( x_{-1} + x_0 > 0 \).

Equation (1) is a special case of equations

\[ x_{n+1} = \frac{Ax_n^2 + Ex_{n-1} + F}{ax_n^2 + ex_{n-1} + f}, \quad n = 0, 1, 2, \ldots \] (2)

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and
\[ x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{ax_n^2 + bx_n x_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f}, \quad n = 0, 1, \ldots \]  

(3)

The dynamics of Equation (2) was investigated in [10] and some special cases were considered in [10, 11]. It was shown that Equation (2) has very rich dynamics ranging from global attractivity of the equilibrium to global period doubling bifurcation to non-conservative chaos. Some special cases of Equation (3) have been considered in the series of papers [8–10, 19, 21].

The special case of Equation (1) when \( A = 1 \) and \( E = 0 \), is the well-known sigmoid Beverton-Holt or Thomson equation
\[ x_{n+1} = \frac{x_n^2}{ax_n^2 + f}, \quad n = 0, 1, \ldots \]  

(4)

which is used in the modelling of fish populations [25].

Notice that Equation (1) is an example of a rational difference equation with an associated map that is always strictly increasing with respect to the second variable and that changes monotonicity with respect to the first variable, i.e. can be increasing or decreasing depending on the corresponding parametric space. There are not many papers that study in detail the dynamics of second order rational difference equations with quadratic terms that have associated maps that change monotonicity with respect to their parameters (see [12, 13, 19]).

Consider the difference equation
\[ x_{n+1} = f(x_n, x_{n-1}), \quad x_{-1}, x_0 \in I, \quad n = 0, 1, \ldots \]  

(5)

where \( f \in C[I \times I, I] \) and \( I \) is some interval of real numbers. Some of our results will be based on the following theorem, see [1, 5].

**Theorem 1.1.** Let \( I \) be a set of real numbers and \( f : I \times I \to I \) be a function which is either non-increasing or non-decreasing in the first variable and non-decreasing in the second variable. Then, for every solution \( \{x_n\}_{n=-1}^\infty \) of Equation (5) the subsequences \( \{x_{2n}\}_{n=0}^\infty \) and \( \{x_{2n-1}\}_{n=0}^\infty \) of even and odd terms of the solution are eventually monotonic sequences.

The consequence of Theorem 1.1 is that every bounded solution of (5) converges to either an equilibrium solution or period-two solution or to the singular point, and the most important question becomes finding the basins of attraction of these solutions as well as any unbounded solutions. The answer to this question follows from the theory of monotone maps in the plane. As we have shown in a sequence of previous papers the boundaries of the basins of attraction of locally asymptotically stable equilibrium solutions or period-two solutions are the global stable manifolds of neighboring saddle points or stable type non-hyperbolic equilibrium solutions or period-two solutions, see [2–4, 16]. The major difference between the cases when \( f(u, v) \) is non-decreasing in \( u \) and when \( f(u, v) \) is non-increasing in \( u \), is the
orientation of the stable manifolds, which are decreasing functions in the first case and increasing functions in the second case. Consequently, one may assume that all solutions of such difference equation will eventually enter an invariant interval where \( f(u, v) \) has specific monotonic behavior with respect to \( u \). In other words, the existence of solutions which oscillate between the regions where the function \( f(u, v) \) changes monotonicity with respect to \( u \) seems to be not feasible. In this paper we give an example of such a case when \( E = f \) and study the global dynamics of that case by using semicycle analysis [14, 15].

By using center manifold theory we investigate the local stability of nonhyperbolic equilibrium points (similar to that in [6, 15, 22, 23]). Our investigation of the global dynamics of Equation (1) makes use of the theory of monotone maps (see [7, 16–18, 20, 24]) since we show that in all cases except \( E = f \) all solutions of Equation (1) will eventually be attracted to an invariant interval where the transition function \( f(u, v) \) is either increasing in both variables or decreasing in the first and increasing in the second variable. As we know from [4, 7, 16] the second iterate \( T^2 \) of the map associated with Equation (5) will be either strongly cooperative or competitive. This fact has been used to obtain several global dynamics results for this type of second order difference equation. For the sake of completeness we list two such results from [7]:

**Theorem 1.2.** Let \( I \subset \mathbb{R} \) be an interval. Consider the Eq.(5) where \( f(x, y) : I \times I \rightarrow I \) is continuous. Suppose that:

(a) \( f \left( \text{int} I \times \text{int} I \right) \subset \text{int} I \), and \( f(x, y) \) is strictly decreasing in \( x \) and strictly increasing in \( y \) in \( \text{int} I \times \text{int} I \).

(b) There exists an equilibrium point \( \bar{x} \in \text{int} I \) and the region of initial conditions \( \left( Q_1(\bar{x}, \bar{x}) \cup Q_2(\bar{x}, \bar{x}) \right) \cap \text{int} I \times \text{int} I \) contains no period-two solutions or equilibria other than \( (\bar{x}, \bar{x}) \). In addition, \( f \) is continuously differentiable on a neighborhood of \( \bar{x} \), and \( \bar{x} \) is a saddle.

Then the global stable manifold of the equilibrium \( W^s(\bar{x}, \bar{x}) \) is a curve in \( I \times I \) which is the graph of a continuous and increasing function that passes through \((\bar{x}, \bar{x})\) and that has endpoints in \( \partial (I \times I) \). Furthermore,

(ii.1) If \( I \) is compact or if there exists a compact set \( K \subset I \) such that \( f(I \times I) \subset K \), then there exist minimal-period two solutions \( (\phi, \psi) \) and \((\psi, \phi)\).

(ii.2) If minimal period two solutions \( (\phi, \psi) \) and \((\psi, \phi)\) exist in \( Q_2(\bar{x}, \bar{x}) \cap I \times I \) and \( Q_4(\bar{x}, \bar{x}) \cap I \times I \) respectively, and if they are the only minimal period-two solutions there, then every solution \( \{x_n\} \) with initial condition in the complement of the global stable manifold of the equilibrium is attracted to one of the period-two solutions. That is, whenever \( x_n \rightarrow \bar{x} \), either \( x_{2n} \rightarrow \phi \) and \( x_{2n+1} \rightarrow \psi \), or \( x_{2n} \rightarrow \psi \) and \( x_{2n+1} \rightarrow \phi \).

(ii.3) If there are no minimal period-two solutions in \( I^2 \), then every solution \( \{x_n\} \) of Eq. (5), with initial condition in \( W^s \) is such that the subsequence \( \{x_{2n}\} \) eventually leaves any given compact subset of \( I \), and every solution \( \{x_n\} \) of Eq.
(5), with initial condition in \(W_+\) is such that the subsequence \(\{x_{2n+1}\}\) eventually leaves any given compact subset of \(I\).

**Theorem 1.3.** Consider Equation (5) where \(f\) is continuous function and \(f\) is decreasing in first argument and increasing in its second argument. Assume that \(\bar{x}\) is a unique equilibrium point which is locally asymptotically stable and assume that \((\phi, \psi)\) and \((\psi, \phi)\) are minimal period-two solutions which are saddle points such that
\[
(\phi, \psi) \leq_{se} (\bar{x}, \bar{x}) \leq_{se} (\psi, \phi).
\]
Then the basin of attraction \(B((\bar{x}, \bar{x}))\) of \((\bar{x}, \bar{x})\) is the region between the global stable manifolds \(W^s((\phi, \psi))\) and \(W^s((\psi, \phi))\). More precisely
\[
B((\bar{x}, \bar{x})) = \{(x, y) : \exists y_u, y_l : y_u < y < y_l, (x, y_1) \in W^s((\phi, \psi)), (x, y_u) \in W^s((\psi, \phi))\}.
\]
The basins of attraction \(B((\phi, \psi)) = W^s((\phi, \psi))\) and \(B((\psi, \phi)) = W^s((\psi, \phi))\) are exactly the global stable manifolds of \((\phi, \psi)\) and \((\psi, \phi)\).

If \((x_{-1}, x_0) \in W^s_+((\psi, \phi))\) or \((x_{-1}, x_0) \in W^s_-((\phi, \psi))\) then \(T^n((x_{-1}, x_0))\) converges to the other equilibrium point or to the other minimal period-two solutions or to the boundary of the region \(I \times I\).

**Remark 1.1.** The analogous results hold in the case where \(f(u, v)\) is increasing in both arguments with the only change being that the orientations of the stable and unstable manifolds are reversed, that is the stable manifolds are decreasing functions while the unstable manifolds are increasing functions, see [3, 4].

**Remark 1.2.** The dynamics of Equation (1) can be described in terms of bifurcation theory as the parameter \(E - f\) pass through several critical values. The first critical value is \(-A^2/4\) where transcritical bifurcation occurs and two additional fixed points \(\bar{x}_- < \bar{x}_+\) appear. The second critical value is 0 where two fixed points \(\bar{x}_-\) and 0 coincide and the zero fixed point changes in local stability from locally asymptotically stable to repeller. The third critical value is \(3A^2/4\) where the locally asymptotically stable period-two point \((\phi, \psi)\) appears and the positive fixed point changes in local stability from locally asymptotically stable to saddle point, giving an example of period-doubling bifurcation. The fourth critical value is \(A^2\) where the period-two point \((\phi, \psi)\) disappears.

### 2. Linearized Stability Analysis

In this section, we present the local stability of the equilibrium points of Equation (1). The equilibrium points of Equation (1) are the positive solutions of the equation
\[
\bar{x} = \frac{A\bar{x} + E\bar{x}}{\bar{x} + f},
\]
or equivalently
\[
\bar{x} (\bar{x}^2 - A\bar{x} - (E - f)) = 0,
\]
from which we obtain
\[
\bar{x}_1 = 0,
\]
Lemma 2.2. Then Equation (1) has a linearized equation

\[ f(u, v) = \frac{Au^2 + Ev}{u^2 + f}, \]

Then Equation (1) has a linearized equation \( z_{n+1} = pz_n + qz_{n-1} \), where

\[ \frac{\partial f}{\partial u} = \frac{2u(Af - vE)}{(u^2 + f)^2}, \quad \frac{\partial f}{\partial v} = \frac{E}{u^2 + f}. \]

**Lemma 2.1.** The equilibrium point \( \bar{x}_1 = 0 \) of Equation (1) is:

i) locally asymptotically stable if \( E < f \),

ii) a repeller if \( E > f \),

iii) a nonhyperbolic point if \( E = f \).

**Proof.** Since

\[ p = \frac{\partial f}{\partial u}(0, 0) = 0, \quad q = \frac{\partial f}{\partial v}(0, 0) = \frac{E}{f}, \]

from the corresponding characteristic equation \( \lambda^2 - p\lambda - q = 0 \) we obtain \( \lambda_{1,2} = \pm \sqrt{\frac{E}{f}} \). Then

\[ |\lambda_{1,2}| \begin{cases} < & 1 \Leftrightarrow E \begin{cases} < & f \end{cases} \end{cases} \]

The partial derivatives at a positive equilibrium point \( \bar{x} \) satisfy:

\[ \begin{cases} p = \frac{\partial f}{\partial u}(\bar{x}, \bar{x}) = \frac{\sqrt{E}(Af - \bar{x}E)}{(\bar{x}^2 + f)^{\frac{3}{2}}}, & \begin{cases} \frac{\partial f}{\partial u}(\bar{x}, \bar{x}) = \frac{\sqrt{E}(Af - \bar{x}E)}{(\bar{x}^2 + f)^{\frac{3}{2}}}, \end{cases} \\
q = \frac{\partial f}{\partial v}(\bar{x}, \bar{x}) = \frac{E}{\bar{x}^2 + f}. & \begin{cases} \frac{\partial f}{\partial v}(\bar{x}, \bar{x}) = \frac{E}{\bar{x}^2 + f}. \end{cases} \end{cases} \]

**Lemma 2.2.** The equilibrium point \( \bar{x}_+ \) of Equation (1) is:

i) a nonhyperbolic point if \( 0 < E = f - \frac{1}{4}A^2 \) (when \( \bar{x}_- = \bar{x}_+ = \frac{A}{2} \)) or \( E = f + \frac{3}{4}A^2 \) (when \( \bar{x}_- = \bar{x}_+ = \frac{3A}{2} \)),

ii) locally asymptotically stable if \( 0 \leq f - \frac{1}{4}A^2 < E < f \) or \( E = f \) or \( f < E < f + \frac{3}{4}A^2 \),

iii) a saddle point if \( f + \frac{3}{4}A^2 < E \).

**Proof.** Notice that \( 0 < 1 - q = \frac{f - E + \bar{x}_+}{f + \bar{x}_+} < 2 \).

i) If \( E = f - \frac{A^2}{4} \) then for \( \bar{x}_+ = \frac{A}{2} \) we have that
This means that the equilibrium point 
which is true because

\[ \lambda_1 = 1, \quad -1 < \lambda_2 = \frac{A^2 - 4f}{A^2 + 4f} < 0. \]

This means that the equilibrium point \( \bar{x}_+ = \frac{A}{2} \) is nonhyperbolic of the stable type. Suppose that \( E = f + \frac{3}{4} A^2 \). Then \( \bar{x}_+ = \frac{3A}{4} \) and

\[ |p| = |1 - q| \Leftrightarrow \bar{x}_+^2 = 3 (E - f) \Leftrightarrow E = f + \frac{3}{4} A^2. \]

\[ ii) \text{ If } 0 \leq f - \frac{1}{4} A^2 < E < f, \text{ then} \]

\[ |p| < 1 - q \Leftrightarrow \frac{2 (f - E)}{f + \bar{x}_+} < \frac{f - E + \bar{x}_+}{f + \bar{x}_+} \Leftrightarrow \bar{x}_+^2 > f - E, \]

which is true because

\[ \bar{x}_+^2 = \left( \frac{A + \sqrt{A^2 + 4 (E - f)}}{2} \right)^2 > \frac{A^2}{4} > f - E. \]

If \( f = E, \) then \( \bar{x}_+ = A \) and

\[ p = \frac{2 (f - E)}{f + \bar{x}_+} = 0, \quad q = \frac{E}{\bar{x}_+ + f} = \frac{E}{A^2 + E}, \]

so that \( 1 - q = 1 - \frac{E}{A^2 + E} = \frac{A^2}{A^2 + E} < 2. \)

Assume that \( f < E < f + \frac{3}{4} A^2. \) Then

\[ |p| < 1 - q \Leftrightarrow \frac{2 (E - f)}{f + \bar{x}_+} < \frac{\bar{x}_+^2 + f - E}{\bar{x}_+ + f} \]

\[ \Leftrightarrow \bar{x}_+^2 > 3 (E - f) \Leftrightarrow A^2 + 2A \sqrt{A^2 + 4 (E - f) + A^2 + 4 (E - f)} > 12 (E - f) \]

\[ \Leftrightarrow A \sqrt{A^2 + 4 (E - f) > 4 (E - f) - A^2 \Leftrightarrow A^4 + 4A^2 (E - f) > (4 (E - f) - A^2)^2 \]

\[ \Leftrightarrow (E - f) (4 f - 4E + 3A^2) > 0 \Leftrightarrow E < f + \frac{3}{4} A^2. \]

\[ iii) \text{ If } f + \frac{3}{4} A^2 < E, \text{ then (similarly as in ii))} \]
The equilibrium points

\( |p| > 1 - q \iff \frac{2(E-f)}{f+x^2_+} > \frac{x^2_+ + f - E}{x^2_+ + f} \)

\( \iff x^2 < 3(E-f) \iff A^2 + 2A\sqrt{A^2 + 4(E-f)} + A^2 + 4(E-f) < 12(E-f) \)

\( \iff 4(E-f)(4f - 4E + 3A^2) < 0 \iff f + \frac{3}{4}A^2 < E. \)

**Lemma 2.3.** The equilibrium point \( x_+ \) of Equation (1):

i) a nonhyperbolic point if \( 0 < E = f - \frac{1}{4}A^2 \) (when \( x_- = x_+ = \frac{4}{7} \)) or \( E = f \) (when \( x_- = x_1 = 0 \)),

ii) a saddle point if \( 0 \leq f - \frac{1}{4}A^2 < E < f, \)

iii) a repeller if \( f < E < f + \frac{3}{4}A^2 \) (when \( x_- = x_1 = 0 \)).

**Proof.** i) This was shown in the proofs of Lemmas 2.1 and 2.2.

ii) Assume that \( 0 \leq f - \frac{1}{4}A^2 < E < f, \)

\(|p| > |1 - q| \iff \frac{2(f-E)}{f+x^2_-} > \frac{f-E + x^2_-}{f+x^2_-} \iff x_- < f - E \)

\( \iff \left(A - \sqrt{A^2 + 4(E-f)}\right)^2 < 4(f-E) \)

\( \iff A^2 + 4(E-f) < A\sqrt{A^2 + 4(E-f)} \)

\( \iff (E-f)(A^2 - 4f + 4E) < 0, \)

which is true when \( 0 \leq f - \frac{1}{4}A^2 < E < f. \)

iii) If \( f < E < f + \frac{3}{4}A^2, \) then \( x_- = 0 \) and is a repeller. This was shown in the proof of Lemma 2.1. \( \square \)

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<th>The equilibrium points</th>
<th>P2-solutions</th>
</tr>
</thead>
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<td>1) ( 0 &lt; E &lt; f - \frac{1}{4}A^2 )</td>
<td>( x_1 = 0 ) (LAS)</td>
<td>-</td>
</tr>
<tr>
<td>2) ( 0 &lt; E = f - \frac{1}{4}A^2 )</td>
<td>( { x_1 = 0 ) (LAS), ( x_2 = \frac{4}{7} ) (NH, ( \lambda_1 = 1, \lambda_2 \in (-1,0) )) }</td>
<td>-</td>
</tr>
<tr>
<td>3) ( 0 \leq f - \frac{1}{4}A^2 &lt; E &lt; f )</td>
<td>( x_1 = 0 ) (LAS), ( x_2 = x_+ ) (SP), ( x_3 = x_+ ) (LAS)</td>
<td>-</td>
</tr>
<tr>
<td>4) ( E = f )</td>
<td>( { x_1 = x_- = 0 ) (NH, ( \lambda_1, \lambda_2 \in \pm 1 )), ( x_2 = x_+ ) (LAS)p }</td>
<td>-</td>
</tr>
<tr>
<td>5) ( f &lt; E &lt; f + \frac{1}{4}A^2 )</td>
<td>( x_1 = x_- = 0 ) (R), ( x_2 = x_+ ) (LAS)p</td>
<td>-</td>
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<tr>
<td>6) ( E = f + \frac{1}{4}A^2 )</td>
<td>( { x_1 = 0 ) (R), ( x_2 = x_+ = \frac{4}{7} ) (NH, ( \lambda_1, \lambda_2 \in (0,1) )) }</td>
<td>-</td>
</tr>
<tr>
<td>7) ( f + \frac{3}{4}A^2 &lt; E &lt; f + A^2 )</td>
<td>( { x_1 = 0 ) (R), ( x_2 = x_+ ) (SP) }</td>
<td>( { \psi, \phi ) (LAS), ( \phi, \psi ) (LAS) }</td>
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<tr>
<td>8) ( E \geq f + A^2 )</td>
<td>( x_1 = 0 ) (R), ( x_2 = x_+ ) (SP)</td>
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</tbody>
</table>

Table 1: Existence and local stability of equilibrium and period-two solutions.
3. Period-Two Solutions

In this section, we investigate the existence and the local stability of the minimal period-two solution of Equation (1).

**Theorem 3.1.** If $f + \frac{3}{2}A^2 < E < f + A^2$, then Equation (1) has the minimal period-two solution
\[
\{\ldots, \phi, \psi, \phi, \psi, \ldots \}, \quad (\phi \neq \psi \text{ and } \phi > 0 \text{ and } \psi > 0),
\]
where
\[
\phi = \frac{(E - f) \left( A - \sqrt{4(E - f) - 3A^2} \right)}{2 (f - E + A^2)}, \quad \psi = \frac{(E - f) \left( A + \sqrt{4(E - f) - 3A^2} \right)}{2 (f - E + A^2)}.
\]

**Proof.** Assume that \{\ldots, \phi, \psi, \phi, \psi, \ldots \} is a minimal period-two solution of (1) with $\phi, \psi \in [0, +\infty)$ and $\phi \neq \psi$. Then
\[
\phi = \frac{A\psi^2 + E\phi}{\psi^2 + f}, \quad \psi = \frac{A\phi^2 + E\psi}{\phi^2 + f},
\]
from which
\[
\phi (\psi^2 + f) = A\psi^2 + E\phi, \quad \psi (\phi^2 + f) = A\phi^2 + E\psi.
\]
By subtracting (12) from (11) we have
\[
(\phi - \psi) (f - \phi\psi - E + A\phi + A\psi) = 0,
\]
i.e.,
\[
f - \phi\psi - E + A(\phi + \psi) = 0. \tag{13}
\]
Similarly, by adding (12) to (11) we obtain
\[
(\phi + \psi) (f + \phi\psi - E - A(\phi + \psi)) + 2A\phi\psi = 0. \tag{14}
\]
If we denote that $\phi + \psi = s (> 0)$ and $\phi\psi = t (> 0)$, then from (13) and (14):
\[
f - E = t - As,
\]
\[
s (f + t) = sE + As^2 - 2At,
\]
which implies that
\[
s \left( f - E + A^2 \right) = A (E - f).
\]
Now,

i) if $f = E$, then $s = t = 0$, which means that there is no minimal period-two solution;

ii) if $f > E$, then $s < 0$, and there is no positive minimal period-two solution;

iii) if $f < E$, then for $E < f + A^2$
\[
s = \frac{A(E - f)}{f - E + A^2} > 0, \quad t = f - E + A \left( \frac{A(E - f)}{f - E + A^2} \right) = \frac{(f - E)^2}{f - E + A^2} > 0,
\]
and
\[ \phi + \psi = \frac{A(E - f)}{f - E + A^2} = s, \quad \phi \psi = \frac{(f - E)^2}{f - E + A^2} = \frac{E - f}s, \]
from which
\[ \phi = \frac{1}{2} s - \frac{1}{2} \sqrt{s - \frac{4(E - f)}{A}} > 0, \quad \psi = \frac{1}{2} s + \frac{1}{2} \sqrt{s - \frac{4(E - f)}{A}} > 0, \]
i.e.,
\[ \phi = \frac{(E - f)(A - \sqrt{4(E - f) - 3A^2})}{2(f - E + A^2)} > 0, \quad \psi = \frac{(E - f)(A + \sqrt{4(E - f) - 3A^2})}{2(f - E + A^2)} > 0, \]
for
\[ f + \frac{3}{4} A^2 < E < f + A^2. \]
\[ \square \]

**Lemma 3.2.** If \( f + \frac{3}{4} A^2 < E < f + A^2 \), then
\[ \psi > \bar{\psi} > \phi > A = \frac{Af}{E}. \]

**Proof.** Indeed i)
\[ \phi = \frac{(E - f)(A - \sqrt{4(E - f) - 3A^2})}{2(f - E + A^2)} > A \]
\[ \Leftrightarrow (E - f)(A - \sqrt{4(E - f) - 3A^2}) > 2A(f - E + A^2) \]
\[ \Leftrightarrow - (E - f)(\sqrt{4(E - f) - 3A^2}) > A(3f - 3E + 2A^2) \]
\[ \Leftrightarrow (E - f)(\sqrt{4(E - f) - 3A^2}) < A(-3f + 3E - 2A^2) \]
\[ \Leftrightarrow (E - f)^2(4(E - f) - 3A^2) < A^2(-3f + 3E - 2A^2)^2 \]
\[ \Leftrightarrow (E - f)^2(4(E - f) - 3A^2) - A^2(-3f + 3E - 2A^2)^2 < 0 \]
\[ \Leftrightarrow - 4(f - E + A^2)^3 < 0, \]
which is true if \( E < f + A^2 \).

ii)
\[ \phi < \bar{\psi} = \frac{A + \sqrt{A^2 + 4(E - f)}}{2} \Leftrightarrow \frac{(E - f)(A - \sqrt{4(E - f) - 3A^2})}{2(f - E + A^2)} < \frac{A + \sqrt{A^2 + 4(E - f)}}{2} \]
\[ \Leftrightarrow (E - f)(A - \sqrt{4(E - f) - 3A^2}) < (f - E + A^2)(A + \sqrt{A^2 + 4(E - f)}) \]
\[
\begin{align*}
&\iff A \left( 2(E-f) - A^2 \right) < \left( f - E + A^2 \right) \sqrt{A^2 + 4(E-f)} + (E-f) \sqrt{4(E-f) - 3A^2} \\
&\iff 4(E-f) - 3A^2 < \sqrt{(A^2 + 4(E-f))(4(E-f) - 3A^2)} \\
&\iff 4A^2 (4f - 4E + 3A^2) < 0,
\end{align*}
\]
which is true because \( f + \frac{3}{4}A^2 < E \).

iii) \[
\bar{x}_n = \frac{A + \sqrt{A^2 + 4(E-f)}}{2} < \psi \iff \frac{A + \sqrt{A^2 + 4(E-f)}}{2} < \frac{(E-f) \left( A + \sqrt{4(E-f) - 3A^2} \right)}{2(f-E + A^2)}
\]
\[
\iff (f-E + A^2) \sqrt{A^2 + 4(E-f)} < A \left( 2(E-f) - A^2 \right) + (E-f) \sqrt{4(E-f) - 3A^2}
\]
\[
\iff 2A^2 (f-E) (4(E-f) - 3A^2) < 2A \left( 2(E-f) - A^2 \right) (E-f) \sqrt{4(E-f) - 3A^2},
\]
which is true because \( f + \frac{3}{4}A^2 < E < f + A^2 \). \qedhere

By the substitution \( x_{n-1} = u_n, \ x_n = v_n \), Equation (1) becomes the system of equations
\[
\begin{aligned}
\begin{cases}
u_{n+1} = v_n \\
v_{n+1} = \frac{A v_n^2 + E u_n}{v_n^2 + f}
\end{cases}
\end{aligned}
\] (15).

The map \( T \) corresponding to Equation (15) is of the form
\[
T \left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} v \\ h(v,u) \end{array} \right),
\] (16)
where \( h(v,u) = \frac{A v^2 + E u}{v^2 + f} \). Since \((\psi, \phi)\) and \((\phi, \psi)\) are the fixed points of the second iteration \( T^2 = T \circ T \) of the map \( T \), i.e.,
\[
T^2 \left( \begin{array}{c} u \\ v \end{array} \right) = T \left( \begin{array}{c} v \\ h(v,u) \end{array} \right) = \left( \begin{array}{c} h(v,u) \\ H(v,u) \end{array} \right),
\] (17)
where
\[
H(v,u) = \frac{A h^2(v,u) + E v}{h^2(v,u) + f},
\]
the Jacobian matrix of the map \( T^2 \) is of the form
\[
J_{T^2}(u,v) = \left( \begin{array}{cc} \frac{\partial h(v,u)}{\partial u} & \frac{\partial h(v,u)}{\partial v} \\ \frac{\partial H(v,u)}{\partial u} & \frac{\partial H(v,u)}{\partial v} \end{array} \right).
\]
Since,
By Lemma 3.2 we have
\[ \frac{\partial h(v,u)}{\partial u} = \frac{E}{v^2 + f}, \quad \frac{\partial h(v,u)}{\partial v} = 2v Af - uE (v^2 + f)^2, \]
\[ \frac{\partial H(v,u)}{\partial u} = \frac{\partial h(v,u)}{\partial u} 2h(v,u) (Af - Ev), \]
\[ \frac{\partial H(v,u)}{\partial v} = \frac{\partial h(v,u)}{\partial v} 2h(v,u) (Af - Ev) \]
\[ (h^2(v,u) + f)^2 + \frac{E}{h^2(v,u) + f}. \]

and
\[ \phi = \frac{A\psi^2 + E\phi}{\psi^2 + f}, \quad \psi = \frac{A\phi^2 + E\psi}{\phi^2 + f}, \quad h(v,u) = \frac{Av^2 + Eu}{v^2 + f}, \]
we obtain
\[ h(\psi,\phi) = \frac{A\psi^2 + E\phi}{\psi^2 + f} = \phi. \]

By Lemma 3.2 we have
\[ \frac{\partial h(\psi,\phi)}{\partial u} = \frac{E}{\psi^2 + f} > 0, \quad \frac{\partial h(\psi,\phi)}{\partial v} = \frac{2\psi (Af - \phi E)}{(\psi^2 + f)^2} < 0, \]
\[ \frac{\partial H(\psi,\phi)}{\partial v} = \frac{\partial h(\psi,\phi)}{\partial v} 2h(\psi,\phi) (Af - Ev) \]
\[ (h^2(\psi,\phi) + f)^2 + \frac{E}{h^2(\psi,\phi) + f}, \]
\[ \frac{\partial H(\psi,\phi)}{\partial v} = \frac{2\psi (Af - \phi E) 2\phi (Af - Ev)}{(\phi^2 + f)^2} + \frac{E}{\phi^2 + f} > 0 \]
\[ \frac{\partial H(\psi,\phi)}{\partial u} = \frac{\partial h(\psi,\phi)}{\partial u} 2h(\psi,\phi) (Af - Ev) \]
\[ (h^2(\psi,\phi) + f)^2 = \frac{E}{\psi^2 + f} \frac{2\phi (Af - Ev)}{(\phi^2 + f)^2} + \frac{E}{\phi^2 + f}. \]

It follows that the Jacobian matrix of the map \( T^2 \) at the point \((\psi,\phi)\) is of the form
\[ J_{T^2} \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} \frac{E}{\psi^2 + f} & \frac{2\phi (Af - Ev)}{(\phi^2 + f)^2} \\ \frac{2\psi (Af - \phi E)}{(\psi^2 + f)^2} & \frac{E}{\phi^2 + f} \end{pmatrix}. \]

The corresponding characteristic equation is
\[ \lambda^2 - p\lambda + q = 0, \]
where
\[ p = \text{Tr} J_{T^2} (\psi,\phi) = \frac{E}{\psi^2 + f} + \frac{2\psi (A - \phi)}{\psi^2 + f} \frac{2\phi (A - \psi)}{\phi^2 + f} + \frac{E}{\phi^2 + f} > 0, \]
\[ q = \text{Det} J_{T^2} (\psi,\phi) = \frac{E^2}{(\psi^2 + f)(\phi^2 + f)} > 0, \]
since \( Af - \phi E = (A - \phi)(\psi^2 + f) \) and \( Af - \psi E = (A - \psi)(\phi^2 + f) \) by Lemma 3.2.
Theorem 3.3. If $f + \frac{3}{4}A^2 < E < f + A^2$, then the minimal period-two solution (9) of Equation (1) is locally asymptotically stable.

Proof. i) By Lemma 3.2 we have that

$$1 + q < 2 \iff \frac{E^2}{(f + \psi^2)} < 1 \iff \frac{E^2}{(f + \psi^2)(f + \phi^2)} < \frac{E^2}{(f + A^2)(f + A^2)} < 1 \iff E^2 - (f + A^2)^2 = (E - f - A^2)(E + f + A^2) < 0,$$

which is true because $E < f + A^2$.

ii) Since

$$p = \frac{E}{\psi^2 + f} + \frac{E}{\phi^2 + f} + \frac{4\phi\psi (Af - \phi E)(Af - E\psi)}{(\psi^2 + f)^2 (\phi^2 + f)^2},$$

$$E \left(2f + (\phi + \psi)^2 - 2\phi\psi\right) + \frac{4\phi\psi (A^2 f^2 + E^2 (\phi + E)(Af - \phi E))}{(\psi^2 + f)^2 (\phi^2 + f)^2}$$

$$= \frac{E}{(f + \psi^2)(f + \phi^2)} + \frac{4\phi\psi (A^2 f^2 + E^2 (\phi + E)(Af - \phi E))}{(\psi^2 + f)^2 (\phi^2 + f)^2}$$

$$= \frac{E}{(f + \psi^2)(f + \phi^2)} + \frac{4\phi\psi (A^2 f^2 + E^2 (\phi + E)(Af - \phi E))}{(\psi^2 + f)^2 (\phi^2 + f)^2}$$

$$= \frac{E}{(f + \psi^2)(f + \phi^2)} + \frac{4\phi\psi (A^2 f^2 + E^2 (\phi + E)(Af - \phi E))}{(\psi^2 + f)^2 (\phi^2 + f)^2}$$

and

$$q = \frac{E^2 (f - E + A^2)^2}{f^2 E^2 + E^4 - 2f E^3 + A^2 f^3 + A^4 f^2 - A^2 f E^2},$$

we obtain

$$p < 1 + q \iff \frac{(f - E)^2 (f - E + A^2)}{f^2 E^2 + E^4 - 2f E^3 + A^2 f^3 + A^4 f^2 - A^2 f E^2} < 0,$$

which is true if

$$f + \frac{3}{4}A^2 < E < f + A^2,$$

because

$$0 < (\phi^2 + f) (\psi^2 + f) = f^2 + f \left( (\phi + \psi)^2 - 2(\phi\psi) \right) + \phi^2 \psi^2$$

$$= f^2 + f \left( \frac{(A(E - f))^2}{f - E + A^2} - 2 \left( \frac{(f - E)^2}{f - E + A^2} \right) \right) + \left( \frac{(f - E)^2}{f - E + A^2} \right)^2$$

$$= \frac{f^2 E^2 + E^4 - 2f E^3 + A^2 f^3 + A^4 f^2 - A^2 f E^2}{(f - E + A^2)^2},$$

i.e.,

$$f^2 E^2 + E^4 - 2f E^3 + A^2 f^3 + A^4 f^2 - A^2 f E^2 = (\psi^2 + f) (\phi^2 + f) (f - E + A^2)^2. \quad \square$$
4. Global asymptotic stability

Notice that the function $f(u, v)$ is always increasing with respect to the second variable, and could be increasing or decreasing with respect to the first variable. The critical point of the function $f(u, v)$ in the first variable is $v = \frac{Af}{E}$.

So, if $v < \frac{Af}{E}$, the function $f(u, v)$ is increasing in the first variable, and if $v > \frac{Af}{E}$, the function $f(u, v)$ is decreasing in the first variable. Since

$$f\left(u, \frac{Af}{E}\right) = \frac{Au^2 + E \left(\frac{Af}{E}\right)}{u^2 + f} = A,$$

(18)

we distinguish the following three cases:

(1) $\frac{Af}{E} = A \iff E = f$,

(2) $\frac{Af}{E} > A \iff E < f$,

(3) $\frac{Af}{E} < A \iff E > f$.

4.1. Case $E = f$

This case corresponds to Case 4 in Table 1, where the zero equilibrium is nonhyperbolic of the resonance type $(-1, 1)$ and the unique positive equilibrium solution is locally asymptotically stable.

If $E = f$, then Equation (1) becomes

$$x_{n+1} = \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + E},$$

(19)

and we obtain the following global result.

**Theorem 4.1.** If $E = f$, then the equilibrium point $\bar{x}_2 = \bar{x}_+ = A$ of Equation (1) is globally asymptotically stable in $(0, \infty)$. More precisely the following statements are true.

(a) If $x_{-1} \geq A$ and $x_0 \geq A$, then $x_n \geq A$ for all $n > 0$ and $\lim_{n \to \infty} x_n = A$.

(b) If $x_{-1} \leq A$ and $x_0 \leq A$, then $x_n \leq A$ for all $n > 0$ and $\lim_{n \to \infty} x_n = A$.

(c) If either $x_{-1} < A < x_0$ or $x_0 < A < x_{-1}$, then $\{x_n\}_{n=-1}^{\infty}$ oscillates about the equilibrium $\bar{x}_2 = \bar{x}_+ = A$ with semicycles of length one and $\lim_{n \to \infty} x_n = A$.

**Proof.** Assume that $(x_0, x_{-1}) \in (0, \infty) \times (0, \infty)$.

If $x_0 \left\{ \begin{array}{ll} \leq A \\
\geq A \end{array} \right.$, then $x_{2k} \left\{ \begin{array}{ll} \leq A \\
\geq A \end{array} \right.$, and if $x_{-1} \left\{ \begin{array}{ll} \leq A \\
\geq A \end{array} \right.$, then $x_{2k+1} \left\{ \begin{array}{ll} \leq A \\
\geq A \end{array} \right.$, for $k = 1, 2, \ldots$. This follows from

$$x_{n+1} - A = \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + E} - A = E \frac{x_{n-1} - A}{x_n^2 + E}.$$
If $x_0 < A$ ($x_0 > A$), then
\[ x_{2k} - x_{2k-2} = \frac{x_{2k-1}^2 (A-x_{2k-2})}{x_{2k-1}^2 + E} > 0 \ (< 0), \]
which means that the sequence $\{x_{2k}\}_{k=0}^\infty$ is increasing (decreasing) and is bounded from above (below) by $A$. Therefore, since there is a unique positive equilibrium solution and there is no a minimal period-two solution, we have
\[ \lim_{k \to \infty} x_{2k} = A. \]
Similarly, if $x_{-1} < A$ ($x_{-1} > A$), then
\[ x_{2k+1} - x_{2k-1} = \frac{x_{2k}^2 (A-x_{2k-1})}{x_{2k}^2 + E} > 0 \ (< 0), \]
which means that the sequence $\{x_{2k+1}\}_{k=0}^\infty$ is increasing (decreasing) and is bounded from above (below) by $A$. Therefore
\[ \lim_{k \to \infty} x_{2k+1} = A. \]
It follows that $\lim_{n \to \infty} x_n = A$ for all $(x_0, x_{-1}) \in (0, \infty) \times (0, \infty)$ (that is $\overline{x}_2 = \overline{x}_+ = A$ is a global attractor) and by Lemma 2.2 the equilibrium point $\overline{x}_2 = \overline{x}_+ = A$ is globally asymptotically stable in $(0, \infty)$.

Remark 4.1. Equation (19) is an example of a difference equation that has solutions which oscillate between two regions $[0, A] \times [A, \infty)$ and $(A, \infty) \times (0, A)$ where the function $f(u, v)$ is increasing and decreasing respectively in $u$ so the theory of monotone maps is inapplicable. Notice that the function $f(u, v)$ is increasing in both variables in $(0, A]^2$ and is decreasing in first and increasing in the second variable in $(A, \infty)^2$.

4.2. Case $E < f$

In this case, we have three qualitatively different situations: 1), 2) and 3) in Table 1.

Lemma 4.2. Suppose that $E < f$. Then an invariant and attracting interval of Equation (1) is $[0, \frac{Af}{E}]$ and the function $f(u, v)$ is increasing in both arguments in this interval.

Proof. Notice that the function $f(u, v)$ is nondecreasing with respect to both variables in $[0, \frac{Af}{E}]^2$. Also, we have
\[ f : \left[0, \frac{Af}{E}\right]^2 \to \left[0, \frac{Af}{E}\right]. \]
which implies that the interval \([0, \frac{Af}{E}]\) is an invariant interval, since
\[
\max_{(x,y) \in [0, \frac{Af}{E}]^2} f(x,y) = f\left(\frac{Af}{E}, \frac{Af}{E}\right) \quad \text{and} \quad \min_{(x,y) \in [0, \frac{Af}{E}]^2} f(x,y) = f(0,0)
\]
and \(f\left(\frac{Af}{E}, \frac{Af}{E}\right) = A < \frac{Af}{E}, f(0,0) = 0\).

Now, we prove that the interval \([0, \frac{Af}{E}]\) is an attracting interval. If \(x_{n-1} > \frac{Af}{E} > A\), then
\[
x_{n+1} - x_{n-1} = \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + f} - x_{n-1} = \frac{x_n^2 (A - x_{n-1}) + x_{n-1} (E - f)}{x_n^2 + f} < 0,
\]
i.e., the sequences \(\{x_{2k}\}_{k=0}^{\infty}\) and \(\{x_{2k+1}\}_{k=0}^{\infty}\) are decreasing, which implies that there exist \(k_0, l_0 \in \mathbb{N}\) such that \(x_{2k} < \frac{Af}{E}\) for \(k \geq k_0\) and \(x_{2k+1} < \frac{Af}{E}\) for \(k \geq l_0\). Otherwise \(x_{2k} \geq \frac{Af}{E}\) and \(x_{2k+1} \geq \frac{Af}{E}\) for all \(k = -1, 0, 1, \ldots\), and \(\lim_{n \to \infty} x_{2k} \geq \frac{Af}{E}\) and \(\lim x_{2k+1} \geq \frac{Af}{E}\), which is a contradiction because there is no minimal period-two solution of Equation (1).

**Theorem 4.3.** Assume that is \(0 < E < f - \frac{4^2}{4}\). Then the unique equilibrium point of Equation (1), \(\bar{x} = 0\), is globally asymptotically stable.

**Proof.** From Lemma 4.2 we see that Equation (1) has only the zero equilibrium point in the invariant and attracting interval \([0, \frac{Af}{E}]\), and the function \(f(u,v)\) is non-decreasing with respect to both variables in \([0, \frac{Af}{E}]^2\). From Theorem 1.4.8 in [14] and Lemma 2.1, we see that the equilibrium point \(\bar{x} = 0\) is globally asymptotically stable. \(\square\)

In the following analysis, we find conditions for local semi-stability of the positive equilibrium point \(\bar{x}_2 = \frac{4}{2}\) of Equation (1), when \(0 < E = f - \frac{4^2}{4}\), using center manifold theory.

**Proposition 4.1.** Assume that \(0 < E = f - \frac{4^2}{4}\). Then the nonhyperbolic equilibrium point \(\bar{x}_2 = \frac{4}{2}\) of Equation (1) is semi-stable from above.

**Proof.** To prove that \(\bar{x}_2\) is semi-stable we will use center manifold theory. Equation (1) is of the form
\[
x_{n+1} = \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + E + \frac{4}{4}A^2}, \quad (20)
\]
By the change of variable \(y_n = x_n - \frac{1}{2}A\), we obtain the following equation (for \(\Omega = 2E + A^2\))
\[
y_{n+1} = \frac{Ay_n^2 + A^2y_n + 2Ey_{n-1}}{2y_n^2 + 2Ay_n + \Omega}, \quad (21)
\]
which has a zero equilibrium. By the substitution \( y_{n-1} = u_n, y_n = v_n \), Equation (21) becomes the system
\[
\begin{align*}
    u_{n+1} &= v_n, \\
    v_{n+1} &= \frac{A v_n^2 + A^2 v_n + 2E v_n + 2E u_n}{2v_n^2 + 2A v_n + \Omega}.
\end{align*}
\] (22)

The Jacobian matrix \( J_0 \) at the zero equilibrium for (22) is
\[
J_0 = \begin{bmatrix}
    0 & 1 \\
    \frac{2E}{\Omega} & \frac{A^2}{\Omega}
\end{bmatrix}
\]
and the corresponding characteristic equation has the form
\[
\lambda^2 - \frac{A^2}{\Omega} \lambda - \frac{2E}{\Omega} = 0,
\]
with
\[
\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = -\frac{2E}{\Omega} \in (-1, 0).
\]
System (22) can be written as
\[
\begin{bmatrix}
    u_{n+1} \\
    v_{n+1}
\end{bmatrix} = J_0 \begin{bmatrix}
    u_n \\
    v_n
\end{bmatrix} + \begin{bmatrix}
    \gamma(u_n, v_n) \\
    \zeta(u_n, v_n)
\end{bmatrix}
\] (23)
where
\[
\begin{align*}
    \gamma(u, v) &= 0, \\
    \zeta(u, v) &= -v \left(2A^2 v^2 + A (A^2 - 2E) v + 4A E u + 4E u v\right) \frac{\Omega}{\Omega (2v^2 + 2A v + \Omega)}.
\end{align*}
\] (24)

Let
\[
\begin{bmatrix}
    u_n \\
    v_n
\end{bmatrix} = P \begin{bmatrix}
    r_n \\
    s_n
\end{bmatrix}
\] (25)
where \( P \) is the matrix that diagonalizes \( J_0 \) defined by
\[
P = \begin{bmatrix}
    1 & 1 \\
    1 & -\frac{2E}{\Omega}
\end{bmatrix},
\]
such that
\[
P^{-1} = -\frac{\Omega}{\Omega + 2E} \begin{bmatrix}
    -\frac{2E}{\Omega} & -1 \\
    -1 & 1
\end{bmatrix},
\]
and
\[
P^{-1} J_0 P = \begin{bmatrix}
    1 & 0 \\
    0 & -\frac{2E}{\Omega}
\end{bmatrix}.
\] (26)
By (25) we have
\[
\begin{align*}
    u_n &= r_n + s_n, \\
    v_n &= r_n - \frac{2E}{\Omega} s_n.
\end{align*}
\]
and by substitution in (24) we have
\[ \gamma(u_n, v_n) = \gamma(r_n, s_n), \]
\[ \zeta(u_n, v_n) = \zeta(r_n, s_n), \]
i.e.,
\[ \gamma(r_n, s_n) = 0, \]
\[ \zeta(r_n, s_n) = \frac{(2E s_n - \Omega r_n) (2 \Omega^3 r_n^2 + A \Omega^3 r_n - 4A^2 E \Omega r_n s_n + 2AE (6E + A^2) \Omega s_n - 16E^3 s_n^2)}{\Omega (\Omega + 2E) (2 \Omega^2 r_n^2 + 2A \Omega^2 r_n - 8E \Omega r_n s_n - 4AE \Omega s_n + 8E^2 s_n^2 + \Omega^3)}. \]
Thus, (23) can be written as
\[ P \begin{bmatrix} r_{n+1} \\ s_{n+1} \end{bmatrix} = J_0 P \begin{bmatrix} r_n \\ s_n \end{bmatrix} + \left[ \begin{array}{c} \gamma(r_n, s_n) \\ \zeta(r_n, s_n) \end{array} \right], \]
or equivalently
\[ \begin{bmatrix} r_{n+1} \\ s_{n+1} \end{bmatrix} = P^{-1} J_0 P \begin{bmatrix} r_n \\ s_n \end{bmatrix} + P^{-1} \left[ \begin{array}{c} \gamma(r_n, s_n) \\ \zeta(r_n, s_n) \end{array} \right]. \]
and by using (26):
\[ \begin{bmatrix} r_{n+1} \\ s_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{2E}{\Omega} \end{bmatrix} \begin{bmatrix} r_n \\ s_n \end{bmatrix} + P^{-1} \left[ \begin{array}{c} \gamma(r_n, s_n) \\ \zeta(r_n, s_n) \end{array} \right]. \] (27)
So, the normal form of System (23) is:
\[ \begin{bmatrix} r_{n+1} \\ s_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{2E}{\Omega} \end{bmatrix} \begin{bmatrix} r_n \\ s_n \end{bmatrix} + \left[ \begin{array}{c} \tilde{\gamma}(r_n, s_n) \\ \tilde{\zeta}(r_n, s_n) \end{array} \right], \] (28)
where
\[ \tilde{\gamma}(r_n, s_n) = -\tilde{\zeta}(r_n, s_n), \]
and
\[ \tilde{\gamma}(r_n, s_n) = \frac{(2E s_n - \Omega r_n) (2 \Omega^3 r_n^2 + A \Omega^3 r_n - 4A^2 E \Omega r_n s_n + 2AE (6E + A^2) \Omega s_n - 16E^3 s_n^2)}{\Omega (\Omega + 2E) (2 \Omega^2 r_n^2 + 2A \Omega^2 r_n - 8E \Omega r_n s_n - 4AE \Omega s_n + 8E^2 s_n^2 + \Omega^3)}. \]
Now, we let \( s = \chi(r) = \Psi(r) + O(r^4) \), where \( \Psi(r) = \alpha r^2 + \beta r^3 \), \( \alpha, \beta \in \mathbb{R} \) is the center manifold, and where the map \( \chi \) must satisfy the center manifold equation (for \( \lambda_2 = -\frac{2E}{\Omega} \)):
\[ \chi(r + \hat{\gamma}(r, \chi(r))) = \lambda_2 \chi(r) - \hat{\zeta}(r, \chi(r)) = 0. \] (29)
If we approximate \( \hat{\gamma}(r, s) \) by a Taylor polynomial as follows
\[ \hat{\gamma}(r, s) = \sum_{i=1}^{3} \frac{1}{i!} \left( r \frac{\partial}{\partial r} y(0, 0) + s \frac{\partial}{\partial s} y(0, 0) \right)^i + O_4, \] (30)
we obtain
\[ \tilde{\gamma}(r, \chi(r)) = r^2 \frac{-A\Omega^2 - 2r\Omega^2 + 2A^2r\Omega - 8A\alpha E^2}{\Omega^2(\Omega + 2E)} + O(r^4), \]
and
\[ \chi'(r + \tilde{\gamma}(r, \chi(r))) = r^2\alpha + \left( \beta - \frac{2A\alpha}{\Omega + 2E} \right)r^3 + O(r^4). \]
Then from (29) we have the following system
\[ 2A \left( 28(2E + A^2)E + 3A^4 \right) \alpha + (2E + A^2)(4E + A^2)^2 \beta = 4(2E + A^2)(E + 2A^2), \]
\[ \alpha (4E + A^2)^2 = A(2E + A^2), \]
with the solution \((\alpha, \beta) = \left( \frac{A(2E + A^2)}{(4E + A^2)^2}, \frac{2(2E + A^2)(16E^2 + 4A^2E + A^4)}{(4E + A^2)^4} \right) \).

Let \( s = \chi(r) = \Psi(r) + O(r^4) \), where
\[ \Psi(r) = A \left( \frac{2E + A^2}{(4E + A^2)^2} \right)r^2 + \frac{2(2E + A^2)(16E^2 + 4A^2E + A^4)}{(4E + A^2)^4}r^3. \]

In view of Theorem 5.9 of [6] the study of the stability of the zero equilibrium of Equation (21), that is the positive nonhyperbolic equilibrium \( \bar{x}_2 = \frac{4}{\alpha} \) of Equation (20), reduces to the stability of the following equation
\[ r_{n+1} = r_n + \tilde{\gamma}(r_n, s_n) = G(r_n), \] (31)
where
\[ G(r) = r + \tilde{\gamma}(r, \Psi(r)) = -r \frac{4E(8E + A^2)r^2 + A(4E + A^2)^2r - (4E + A^2)^3}{(A^2 + 4E)^3}. \]

Since \( \frac{d}{dr} G(0) = 1 \) and
\[ \frac{d^2}{dr^2} G(0) = -\frac{2A}{A^2 + 4E} < 0, \]
from Theorem 1.6 of [15], the zero equilibrium of (31) is an unstable fixed point, that is semi-stable from above. Therefore, from Theorem 5.9 of [6], the zero equilibrium of Equation (21), that is the positive nonhyperbolic equilibrium \( \bar{x}_2 = \frac{4}{\alpha} \) of Equation (20) is semi-stable from above. \( \Box \)

The next result shows global behavior of Equation (1) when \( 0 < E = f - \frac{A^2}{4} \).

**Theorem 4.4.** Assume that \( 0 < E = f - \frac{A^2}{4} \). Then Equation (1) has two equilibrium points: \( \bar{x}_1 = 0 \) which is locally asymptotically stable and \( \bar{x}_2 = \frac{4}{\alpha} \) which is nonhyperbolic of the stable type, more precisely \( \bar{x}_2 \) is semi-stable. There exists a set \( C \subset Q_2(\bar{x}_2, \bar{x}_2) \cup Q_4(\bar{x}_2, \bar{x}_2) \) with endpoints on the axes and \( W^s((\bar{x}_2, \bar{x}_2)) = C \) is an invariant subset of the basin of attraction of \((\bar{x}_2, \bar{x}_2)\). Also, \( C \) is a graph of a
strictly decreasing continuous function of the first variable on an interval and separates $R = [0, +\infty)^2$ into two connected and invariant components $W_-( (\bar{x}_2, \bar{x}_2))$ and $W_+ ( (\bar{x}_2, \bar{x}_2))$, where

$$W_- ( (\bar{x}_2, \bar{x}_2)) := \{(x, y) \in R \setminus C : \exists (x', y') \in C \text{ with } (x, y) \preceq_{ne} (x', y')\}$$

and

$$W_+ ( (\bar{x}_2, \bar{x}_2)) := \{(x, y) \in R \setminus C : \exists (x', y') \in C \text{ with } (x', y') \preceq_{ne} (x, y)\},$$

such that:

(i) if $(x-1, x_0) \in W_+ ( (\bar{x}_2, \bar{x}_2))$, then $\lim_{n \to \infty} x_n = \frac{4}{2}$;

(ii) if $(x-1, x_0) \in W_- ( (\bar{x}_2, \bar{x}_2))$, then $\lim_{n \to \infty} x_n = 0$.

**Proof.** Since

$$x_{n+1} - A = \frac{A}{2} (x_n - \frac{A}{2}) (x_n + \frac{A}{2}) + E (x_n - \frac{A}{2}) x_n + E + \frac{A^2}{4},$$

this means that $[0, \frac{A}{2}]$ and $[\frac{A}{2}, +\infty)$ are invariant intervals, i.e., if $(x-1, x_0) \in [0, \frac{A}{2}]^2$ (or $[\frac{A}{2}, +\infty)^2$), then $x_n \in [0, \frac{A}{2}]$ (or $[\frac{A}{2}, +\infty)$ for all $n = 1, 2,...$.

By using Lemma 4.2, Proposition 4.1 and the theory of monotone maps in the plane, more precisely, the theory of cooperative maps, since the corresponding map $T^2$ from (17) is a cooperative map, the conclusion of the theorem follows. In other words, the version of Theorem 1.2 for a function increasing in both variables applies. $\square$

**Theorem 4.5.** Assume that $0 < f - \frac{A^2}{4} < E < f$. Then Equation (1) has three equilibrium points: $\bar{x}_1 = 0$ and $\bar{x}_3 = \bar{x}_4$ which are locally asymptotically stable and $\bar{x}_2 = \bar{x}_-$ which is a saddle point. There exists a set $C \subset Q_2 (\bar{x}_2, \bar{x}_2) \cup Q_3 (\bar{x}_-, \bar{x}_2)$ with endpoints on the axes and $W_s ( (\bar{x}_2, \bar{x}_2)) = C$ is an invariant subset of the basin of attraction of $(\bar{x}_2, \bar{x}_2)$. Also, $C$ is a graph of a strictly decreasing continuous function of the first variable on an interval and separates $R = [0, +\infty)^2$ into two connected and invariant components $W_-( (\bar{x}_2, \bar{x}_2))$ and $W_+ ( (\bar{x}_2, \bar{x}_2))$ such that:

(i) if $(x-1, x_0) \in W_+ ( (\bar{x}_2, \bar{x}_2))$, then $\lim_{n \to \infty} x_n = \bar{x}_+$;

(ii) if $(x-1, x_0) \in W_- ( (\bar{x}_2, \bar{x}_2))$, then $\lim_{n \to \infty} x_n = 0$.

**Proof.** It follows from Lemmas 2.2 and 4.2 and the theory of monotone maps in the plane, more precisely theory of cooperative maps, since the corresponding map $T^2$ is a cooperative map. $\square$

**4.3. Case $E > f$**

In this case, we have four qualitatively different situations: 5), 6), 7) and 8) in Table 1.
Lemma 4.6. If $f < E < f + A^2$ then an invariant and attracting interval of Equation (1) is $\left[A, \frac{A^3}{f + A^2 - E}\right]$ and the function $f(u, v)$ is decreasing in the first and increasing in the second argument in this interval.

Proof. If $v \geq A > \frac{Af}{E}$, then the function $f(u, v)$ is decreasing in the first variable and increasing in the second variable, which implies that

$$x_{n+1} = \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + f} > \frac{Ax_n^2 + E\frac{Af}{E}}{x_n^2 + f} = A.$$  

Therefore, if there exists an invariant interval of Equation (1), then it is of the form $[A, U]$. Since, the function $f$ is decreasing in the first variable and increasing in the second variable in $[A, U]^2$, we obtain that

$$U \geq \max_{(x, y) \in [A, U]^2} f(x, y) = f(A, U)$$ and $$\min_{(x, y) \in [A, U]^2} f(x, y) = f(U, A) \geq A,$$

i.e.,

$$U \geq f(A, U) \iff U \geq \frac{A^3 + EU}{A^2 + f} \iff \left(U \geq \frac{A^3}{f + A^2 - E} \text{ and } E < f + A^2\right),$$

from which we can set $U = \frac{A^3}{f + A^2 - E}$ when $E < f + A^2$. This means that $\left[A, \frac{A^3}{f + A^2 - E}\right]$ is an invariant interval of Equation (1) when $E < f + A^2$.

On the other hand, if $x_{n-1} < \frac{Af}{E} < A$, we obtain

$$x_{n+1} - x_n - x_{n-1} = \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + f} - x_{n-1} = \frac{x_n^2(A - x_{n-1}) + x_{n-1}(E - f)}{x_n^2 + f} > 0,$$

i.e., the sequences $\{x_{2k}\}_{k=0}^{\infty}$ and $\{x_{2k+1}\}_{k=0}^{\infty}$ are increasing, which implies that there exist $k_0, l_0 \in \mathbb{N}$ such that $x_{2k} > \frac{Af}{E}$ for $k \geq k_0$ and $x_{2k+1} > \frac{Af}{E}$ for $k \geq l_0$. Otherwise $x_{2k} \leq \frac{Af}{E}$ and $x_{2k+1} \leq \frac{Af}{E}$ for all $k = -1, 0, 1, \ldots$, and $\lim_{n \to \infty} x_{2k} = \frac{Af}{E}$ and $\lim_{n \to \infty} x_{2k+1} = \frac{Af}{E}$, which is a contradiction because the zero equilibrium is a repeller and there is no a minimal period-two solution of Equation (1) in $\left[0, \frac{Af}{E}\right]$. This means that $\left[A, \frac{A^3}{f + A^2 - E}\right]$ is also an attracting interval for Equation (1) when $E < f + A^2$. □

Theorem 4.7. Assume that $f < E < f + \frac{3}{4} A^2$. Then Equation (1) has two equilibrium points: $\overline{x}_1 = 0$ which is a repeller and $\overline{x}_2 = \overline{x}_+ = \overline{x}$ which is locally asymptotically stable. Also, the positive equilibrium point $\overline{x}_2 = \overline{x}_+$ is globally asymptotically stable in $(0, +\infty)$.

Proof. The function $f(u, v)$ is decreasing in the first variable and increasing in the second variable in $\left[A, \frac{A^3}{f + A^2 - E}\right]$ and there is no a minimal period-two solution. Since $\left[A, \frac{A^3}{f + A^2 - E}\right]$ is an invariant and an attracting interval of Equation (1), using
Theorem 1.4.6 in [14] and Lemma 2.2, we see that the positive equilibrium point \( x_2 = \bar{x}_+ \) is globally asymptotically stable in \((0, +\infty)\). \(\square\)

Now, by using center manifold theory we find conditions for the stability of the positive equilibrium point \( x_2 = \bar{x}_+ = \frac{3}{2}A \) of Equation (1), when \( E = f + \frac{3}{4}A^2 \).

**Proposition 4.2.** Assume that \( E = f + \frac{3}{4}A^2 \). Then the positive equilibrium point \( x_2 = \bar{x}_+ = \frac{3}{2}A \) of Equation (1) is locally asymptotically stable.

**Proof.** To prove that \( x_2 \) is a local sink we will use center manifold theory. Since \( E = f + \frac{3}{4}A^2 \), Equation (1) is of the form

\[
x_{n+1} = \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + E - \frac{3}{4}A^2}.
\]

By the change of variable \( y_n = x_n - \frac{3}{2}A \), we get the equation (for \( \Phi = 3A^2 + 2E \))

\[
y_{n+1} = \frac{-Ay_n^2 - 3A^2y_n + 2Ey_{n-1}}{2y_n^2 + 6Ay_n + \Phi},
\]

which has a zero equilibrium. By the substitution \( y_{n-1} = u_n, y_n = v_n \), Equation (33) becomes the system

\[
\begin{align*}
  u_{n+1} &= v_n, \\
  v_{n+1} &= \frac{-Ay_n^2 - 3A^2v_n + 2Ey_{n-1}}{2y_n^2 + 6Ay_n + \Phi}.
\end{align*}
\]

The Jacobian matrix \( J_0 \) at the zero equilibrium for (34) is

\[
J_0 = \begin{bmatrix}
  0 & \frac{3A^2}{\Phi} \\
  \frac{2E}{\Phi} & 1
\end{bmatrix}
\]

and the corresponding characteristic equation is

\[
\lambda^2 + \frac{3A^2}{\Phi} \lambda - \frac{2E}{\Phi} = 0,
\]

with

\[
\lambda_1 = -1, \lambda_2 = \frac{2E}{\Phi} \in (0, 1).
\]

System (34) can be written as

\[
\begin{bmatrix}
  u_{n+1} \\
  v_{n+1}
\end{bmatrix} = J_0 \begin{bmatrix}
  u_n \\
  v_n
\end{bmatrix} + \begin{bmatrix}
  \gamma(u_n, v_n) \\
  \zeta(u_n, v_n)
\end{bmatrix}
\]

where

\[
\gamma(u, v) = 0,
\]

\[
\zeta(u, v) = v \frac{18A^3v + 6A^2v^2 - Av\Phi - 12AuE - 4uvE}{\Phi(\Phi + 6Av + 2v^2)}.
\]

Let

\[
\begin{bmatrix}
  u_n \\
  v_n
\end{bmatrix} = P \begin{bmatrix}
  r_n \\
  s_n
\end{bmatrix},
\]

(37)
where $P$ is the matrix that diagonalizes $J_0$ defined by

$$
P = \begin{bmatrix}
1 & 1 \\
-1 & \frac{2E}{\Phi}
\end{bmatrix}.
$$

Then

$$
P^{-1} = \frac{\Phi}{\Phi + 2E} \begin{bmatrix}
\frac{2E}{\Phi} & -1 \\
1 & 1
\end{bmatrix},
$$

and

$$
P^{-1}J_0P = \begin{bmatrix}
-1 & 0 \\
0 & \frac{2E}{\Phi}
\end{bmatrix}.
$$

The normal form of system (35), obtained in a similar manner as in the proof of Proposition 4.1, is

$$
\begin{align*}
\begin{bmatrix}
r_{n+1} \\
s_{n+1}
\end{bmatrix} &= \begin{bmatrix}
-1 & 0 \\
0 & \frac{2E}{\Phi}
\end{bmatrix} \begin{bmatrix}
r_n \\
s_n
\end{bmatrix} + \begin{bmatrix}
\hat{\gamma}(r_n,s_n) \\
\hat{\zeta}(r_n,s_n)
\end{bmatrix},
\end{align*}
$$

where

$$
\hat{\gamma}(r_n,s_n) = -\hat{\zeta}(r_n,s_n),
$$

and

$$
\hat{\gamma}(r_n,s_n) = -\frac{\Phi r_n - 2Es_n - 2\Phi^3 r_n^2 + 16E^3 s_n + 5A\Phi^3 r_n + 2A\Phi E(\Phi + 12)s_n + 12A^2\Phi Er_n s_n}{2\Phi^2 r_n^2 + 8E^2 s_n^2 - 6A\Phi^2 r_n + 12A\Phi Es_n - 8E^2 r_n s_n + \Phi^3}.
$$

Now, let $s = \chi(r) = \Psi(r) + O(r^4)$, where $\Psi(r) = \alpha r^2 + \beta r^3$, $\alpha, \beta \in \mathbb{R}$ is the center manifold, and where map $\chi$ must satisfy the center manifold equation (for $\lambda_2 = \frac{2E}{\Phi}$)

$$
\chi(-r + \hat{\gamma}(r,\chi(r))) - \lambda_2 \chi(r) - \hat{\zeta}(r,\chi(r)) = 0.
$$

By (30), we have that

$$
\hat{\gamma}(r,\chi(r)) = -\frac{5A\Phi^2 r^2 + 2(\Phi(-\Phi + 15A^2) - 4A\alpha E(\Phi - 3E)) r^3}{\Phi^2(\Phi + 2E)} + O(r^4),
$$

and

$$
\chi(-r + \hat{\gamma}(r,\chi(r))) = \frac{\alpha(\Phi + 2E)r^2 + (10A\alpha - (\Phi + 2E)\beta)r^3}{\Phi + 2E} + O(r^4).
$$

Then from (40) we have the system

$$
2A(6E - 5\Phi)(\Phi + 2E)\alpha + \Phi(\Phi + 2E)^2\beta = 2\Phi(\Phi - 15A^2),
$$

$$
\alpha(\Phi^2 - 4E^2) = 5A\Phi,
$$

with the solution $(\alpha, \beta) = \left(\frac{5}{3A(4E + 3A^2)}, \frac{26}{3}(\frac{2E + 3A^2}{(4E + 3A^2)^2})\right)$.

Let $s = \chi(r) = \Psi(r) + O(r^4)$, where

$$
\Psi(r) = \frac{5}{3A(4E + 3A^2)}r^2 + \frac{26}{3}(\frac{2E + 3A^2}{(4E + 3A^2)^2})r^3.
$$
Now, according to Theorem 5.9 of [6] the study of the stability of the zero equilibrium of Equation (33), that is the positive nonhyperbolic equilibrium $\bar{x}_2 = \frac{3}{2}A$ of Equation (32), reduces to the study of stability of the following equation

$$r_{n+1} = -r_n + \tilde{\gamma}(r_n,s_n) = G(r_n), \quad (41)$$

where

$$G(r) = -r + \tilde{\gamma}(r,\Psi(r)) = \frac{r \left(4(-E+18A^2)r^2+15A(4E+3A^2)r+3(4E+3A^2)^2\right)}{3(4A^2+4E)^2}.$$ 

Since $\frac{d}{dr}G(0) = -1$, $\frac{d}{dr}G(0) = -\frac{10A}{3A^2+4E}$ and $\frac{d^3}{dr^3}G(0) = -\frac{8(18A^2-E)}{(3A^2+4E)^2}$, then the corresponding Schwarzian is of the form

$$SG(0) = -\frac{d^3}{dr^3}G(0) - \frac{3}{2} \left(\frac{d^2}{dr^2}G(0)\right)^2 = -\frac{2}{3A^2+4E} < 0,$$

and from Theorem 1.6 of [15], the zero equilibrium of (41) is a sink. Therefore, from Theorem 5.9 of [6], the zero equilibrium of Equation (33), that is the positive nonhyperbolic equilibrium $\bar{x}_2 = \frac{3}{2}A$ of Equation (32) is a sink. \hfill \Box

The next result shows global behavior of Equation (1) when $E = f + \frac{3}{4}A^2$.

**Theorem 4.8.** If $E = f + \frac{3}{4}A^2$, then the positive equilibrium point $\bar{x}_2 = \bar{x}_+ = \frac{3}{2}A$ is globally asymptotically stable in $(0, +\infty)$.

**Proof.** The proof is the same as the proof of Theorem 4.7 since the equilibrium point $\bar{x}_2$ is stable (see Proposition 4.2). \hfill \Box

**Theorem 4.9.** Assume that $f + \frac{3}{4}A^2 < E < f + A^2$. Then Equation (1) has two equilibrium points: $\bar{x}_1 = 0$, which is a repellor and $\bar{x}_2 = \bar{x}_+$, which is a saddle point, and has the unique minimal period-two solution $\{...\phi,\psi,\phi,\psi,...\}$, which is locally asymptotically stable, where $\phi$ and $\psi$ are of the form (10). There exists a set $\mathcal{C} \subset Q_1(\bar{x}_2,\bar{x}_2) \cup Q_2(\bar{x}_2,\bar{x}_2)$ and $\mathcal{W}^s((\bar{x}_2,\bar{x}_2)) = \mathcal{C}$ is the basin of attraction of $(\bar{x}_2,\bar{x}_2)$. The set $\mathcal{C}$ is a graph of a strictly increasing continuous function of the first variable on an interval and separates $\mathcal{R}_1 = [0, \infty)^2 \setminus \{(0,0)\}$ into two connected and invariant parts, $\mathcal{W}_-^{(1)}((\bar{x}_2,\bar{x}_2))$ and $\mathcal{W}_+^{(1)}((\bar{x}_2,\bar{x}_2))$, where

$$\mathcal{W}_-^{(1)}((\bar{x}_2,\bar{x}_2)) : = \{(x,y) \in \mathcal{R}_1 \setminus \mathcal{C} : \exists (x',y') \in \mathcal{C} \text{ with } (x,y) \preceq_{se} (x',y')\}$$

$$\mathcal{W}_+^{(1)}((\bar{x}_2,\bar{x}_2)) : = \{(x,y) \in \mathcal{R}_1 \setminus \mathcal{C} : \exists (x',y') \in \mathcal{C} \text{ with } (x',y') \preceq_{se} (x,y)\},$$

such that:

(i) if $(x_{-1},x_0) \in \mathcal{W}_+^{(1)}((\bar{x}_2,\bar{x}_2))$, then $\lim_{n \to \infty} x_{2n} = \phi$ and $\lim_{n \to \infty} x_{2n+1} = \psi$;

(ii) if $(x_{-1},x_0) \in \mathcal{W}_-^{(1)}((\bar{x}_2,\bar{x}_2))$, then $\lim_{n \to \infty} x_{2n} = \psi$ and $\lim_{n \to \infty} x_{2n+1} = \phi$. 


Theorem 4.11. More precisely, competitive maps, since the corresponding map \( T^2 \) is a competitive map, see [7,16]. In other words Theorem 1.3 applies.

Lemma 4.10. Suppose that \( E \geq f + A^2 \). Then an invariant and attracting interval of Equation (1) is \([A, +\infty)\).

Proof. If \( x_{n-1} > A \left( > \frac{Af}{E} \right) \), then
\[
x_{n+1} = \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + f} > \frac{Ax_n^2 + EA}{x_n^2 + f} > \frac{Ax_n^2 + E\frac{Af}{E}}{x_n^2 + f} = A > \frac{Af}{E},
\]
so that
\[
x_{n-1}, x_n \in [A, +\infty) \Rightarrow x_{n+1} \in [A, +\infty),
\]
i.e., \([A, +\infty)\) is an invariant interval for Equation (1). The proof of the fact that \([A, +\infty)\) is also an attracting interval of Equation (1) is the same as in Lemma 4.6.

Theorem 4.11. If \( E \geq f + A^2 \), then Equation (1) has two equilibrium points: \( \bar{x}_1 = 0 \), which is a repeller and \( \bar{x}_2 = \bar{x}_+ \), which is a saddle point. There exists a set \( C \subset Q_1(\bar{x}_2, \bar{x}_2) \cup Q_3(\bar{x}_2, \bar{x}_2) \) and \( W^b((\bar{x}_2, \bar{x}_2)) = C \) is the basin of attraction of \((\bar{x}_2, \bar{x}_2)\). The set \( C \) is a graph of a strictly increasing continuous function of the first variable on an interval and separates \( \mathcal{R}_1 = [0, \infty)^2 \setminus \{(0,0)\} \) into two connected and invariant parts, \( W^{(1)}_-(\bar{x}_2, \bar{x}_2) \) and \( W^{(1)}_+(\bar{x}_2, \bar{x}_2) \), such that:

(i) if \((x_{-1}, x_0) \in W^{(1)}_-(\bar{x}_2, \bar{x}_2)\), then \( \lim_{n \to \infty} x_{2n+1} = +\infty \) and \( \lim_{n \to \infty} x_{2n} = A \);

(ii) if \((x_{-1}, x_0) \in W^{(1)}_+(\bar{x}_2, \bar{x}_2)\), then \( \lim_{n \to \infty} x_{2n+1} = A \) and \( \lim_{n \to \infty} x_{2n} = +\infty \).

Proof. By using Lemma 4.10 it follows that every solution of Equation (1) eventually enters the interval \([A, +\infty)\). Since then the function \( f(x_n, x_{n-1}) \) is decreasing in the first variable and increasing in the second variable in \([A, +\infty)\), we can apply Theorem 1.1 and the theory of monotone maps, the corresponding map \( T^2 \) is a competitive. More precisely Theorem 1.3 applies. It is easy to see that if \( \lim_{n \to \infty} x_{2n+1} = +\infty \), then \( \lim_{n \to \infty} x_{2n} = l < \infty \), which implies
\[
l = \lim_{n \to \infty} x_{2n+2} = \lim_{n \to \infty} A + \frac{E x_{2n}}{x_{2n+1}} = A.
\]
Also, if \( \lim_{n \to \infty} x_{2n} = +\infty \) then \( \lim_{n \to \infty} x_{2n+1} = A \).

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REFERENCES


GLOBAL DYNAMICS OF CERTAIN NON-SYMMETRIC SECOND ORDER DIFFERENCE EQUATION WITH QUADRATIC TERM

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Dedicated to the memory of Professor Harry Miller and Professor Fikret Vajzović

ABSTRACT. We investigate global dynamics of the equation

\[ x_{n+1} = \frac{x_{n-1} + F}{ax_n^2 + f}, \quad n = 0, 1, 2, \ldots, \]

where the parameters \( a, F \) and \( f \) are positive numbers and the initial conditions \( x_{-1}, x_0 \) are arbitrary nonnegative numbers such that \( x_{-1} + x_0 > 0 \). The existence and local stability of the unique positive equilibrium are analyzed algebraically. We characterize the global dynamics of this equation with the basins of attraction of its equilibrium point and periodic solutions.

1. INTRODUCTION AND PRELIMINARIES

We investigate global behavior of the equation

\[ x_{n+1} = \frac{x_{n-1} + F}{ax_n^2 + f}, \quad n = 0, 1, 2, \ldots, \tag{1.1} \]

where parameters \( a, f \) and \( F \) are positive numbers and the initial conditions \( x_{-1}, x_0 \) are arbitrary nonnegative numbers. For Equation (1.1) we will precisely define the basins of attraction of all attractors, which consist of the equilibrium point, period-two solution and points at infinity. The special case of Equation (1.1), where \( F = 0 \),

\[ x_{n+1} = \frac{x_{n-1}}{ax_n^2 + f} \tag{1.2} \]

were studied in detail in [8]. The presence of parameter \( F \) in the Equation (1.1) excludes a scenario of coexistence of infinite number nonhyperbolic minimal-period two solutions which is possible in Equation (1.2) for some values of parameters. Both equations, (1.1) and (1.2), are special cases of equation

\[ x_{n+1} = \frac{Ax_n^2 + Ex_{n-1} + F}{ax_n^2 + ex_{n-1} + f}, \quad n = 0, 1, 2, \ldots \tag{1.3} \]

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which was considered in [7]. The global asymptotic stability results were obtained in [7] for several special cases of Equation (1.3), where the right-hand side does not change its monotonicity. Some special second order quadratic fractional difference equations have been considered in the series of papers, see [1, 2, 5, 6, 11, 12, 16, 17]. Also, several global asymptotic results for some special cases of a general second order quadratic fractional difference equation were obtained in [9,10]. Our investigation of the global character of Equation (1.1) will be based on the theory of competitive systems and difference inequalities.

We will use the following theorem for a general second order difference equation

\[ x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, 2, \ldots, \] (1.4)

see [4].

**Theorem 1.1.** Let \([a, b]\) be an interval of real numbers and assume that \(f : [a, b] \times [a, b] \to [a, b]\) is a continuous function satisfying the following properties:

(a) \(f(x, y)\) is non-increasing in first and non-decreasing in second variable.

(b) Equation (1.4) has no minimal period-two solutions in \([a, b]\).

Then every solution of Equation (1.4) converges to \(\bar{x}\).

**Theorem 1.2.** Let \(T\) be a competitive map on a rectangular region \(\mathcal{R} \subset \mathbb{R}^2\). Let \(\bar{x} \in \mathcal{R}\) be a fixed point of \(T\) such that \(\Delta := \mathcal{R} \cap \text{int} (Q_1(\bar{x}) \cup Q_3(\bar{x}))\) is nonempty (i.e., \(\bar{x}\) is not the NW or SE vertex of \(\mathcal{R}\)), and \(T\) is strongly competitive on \(\Delta\). Suppose that the following statements are true.

a. The map \(T\) has a \(C^1\) extension to a neighborhood of \(\bar{x}\).

b. The Jacobian \(J_T(\bar{x})\) of \(T\) at \(\bar{x}\) has real eigenvalues \(\lambda, \mu\) such that \(0 < |\lambda| < \mu\), where \(|\lambda| < 1\), and the eigenspace \(E^\lambda\) associated with \(\lambda\) is not a coordinate axis.

Then there exists a curve \(C \subset \mathcal{R}\) through \(\bar{x}\) that is invariant and a subset of the basin of attraction of \(\bar{x}\), such that \(C\) is tangential to the eigenspace \(E^\lambda\) at \(\bar{x}\), and \(C\) is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of \(C\) in the interior of \(\mathcal{R}\) are either fixed points or minimal period-two points. In the latter case, the set of endpoints of \(C\) is a minimal period-two orbit of \(T\).

For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 1.2 reduces just to \(|\lambda| < 1\). Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis.

**Theorem 1.3.** (A) Assume the hypotheses of Theorem 1.2, and let \(C\) be the curve whose existence is guaranteed by Theorem 1.2. If the endpoints of \(C\) belong to \(\partial \mathcal{R}\), then \(C\) separates \(\mathcal{R}\) into two connected components, namely

\[ W_- := \left\{ x \in \mathcal{R} \setminus C : \exists y \in C \text{ with } x \preceq_{se} y \right\} \] and

\[ W_+ := \left\{ x \in \mathcal{R} \setminus C : \exists y \in C \text{ with } y \preceq_{se} x \right\}, \] (1.5)

such that the following statements are true.
(i) $W_-$ is invariant, and dist$(T^n(x), Q_2(\overline{x})) \to 0$ as $n \to \infty$ for every $x \in W_-$. 
(ii) $W_+$ is invariant, and dist$(T^n(x), Q_4(\overline{x})) \to 0$ as $n \to \infty$ for every $x \in W_+$. 
(B) If, in addition to the hypotheses of part (A), $\overline{x}$ is an interior point of $\mathcal{R}$ and $T$ is $C^2$ and strongly competitive in a neighborhood of $\overline{x}$, then $T$ has no periodic points in the boundary of $Q_1(\overline{x}) \cup Q_3(\overline{x})$ except for $\overline{x}$, and the following statements are true. 
(ii) For every $x \in W_-$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int} Q_2(\overline{x})$ for $n \geq n_0$. 
(iv) For every $x \in W_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int} Q_4(\overline{x})$ for $n \geq n_0$. 

If $T$ is a map on a set $\mathcal{R}$ and if $\overline{x}$ is a fixed point of $T$, the stable set $W^s(\overline{x})$ of $\overline{x}$ is the set $\{x \in \mathcal{R} : T^n(x) \to \overline{x}\}$ and unstable set $W^u(\overline{x})$ of $\overline{x}$ is the set 
\[
\{ x \in \mathcal{R} : \text{there exists } \{x_n\}_{n=-\infty}^{0} \subset \mathcal{R} \text{ s.t. } T(x_n) = x_{n+1}, x_0 = x, \text{ and } \lim_{n \to -\infty} x_n = \overline{x} \}.
\]

When $T$ is non-invertible, the set $W^s(\overline{x})$ may not be connected and made up of infinitely many curves, or $W^u(\overline{x})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on $\mathcal{R}$, the sets $W^s(\overline{x})$ and $W^u(\overline{x})$ are actually the global stable and unstable manifolds of $\overline{x}$.

**Remark 1.1.** We say that $f(u,v)$ is strongly decreasing in the first argument and strongly increasing in the second argument if it is differentiable and has first partial derivative $D_1 f$ negative and first partial derivative $D_2 f$ positive in a considered set. The connection between the theory of monotone maps and the asymptotic behavior of Equation (1.4) follows from the fact that if $f$ is strongly decreasing in the first argument and strongly increasing in the second argument, then the second iterate of a map associated to Equation (1.4) is a strictly competitive map on $I \times I$, see [14].

Next result is one of the basic results on difference equation inequalities which we will use in this paper.

**Theorem 1.4.** [3] Let $n \in N_{n_0}^{+}$ and $g(n,u,v)$ be a nondecreasing function in $u$ and $v$ for any fixed $n$. Suppose that for $n \geq n_0$, the inequalities 
\[
y_{n+1} \leq g(n,y_n,y_{n-1}) \quad (1.6)
\]
\[
u_{n+1} \geq g(n,u_n,u_{n-1}) \quad (1.7)
\]
hold. Then 
\[
y_{n_0-1} \leq u_{n_0-1}, \quad y_{n_0} \leq u_{n_0}
\]
implies that 
\[
y_n \leq u_n \quad n \geq n_0.
\]

The rest of this paper is organized as follows. The second section presents the local stability of the unique positive equilibrium solution. The third section gives conditions for existence of the minimal period-two solution and its local stability.
The fourth section presents global dynamics in certain regions of the parametric space.

2. LOCAL STABILITY ANALYSIS

In this section, we present the local stability of the unique positive equilibrium of Equation (1.1). The equilibrium points of Equation (1.1) are the positive solutions of the equation

$$\bar{x} = \frac{\bar{x} + F}{a\bar{x}^2 + f}$$

i.e.

$$a\bar{x}^3 + (f - 1)\bar{x} - F = 0. \quad (2.1)$$

We will denote the left side of the previous relation by

$$G(x) = ax^3 + (f - 1)x - F.$$ 

Then it holds

$$G'(x) = 3ax^2 + f - 1 \quad \text{and} \quad G'(x) = 0 \Leftrightarrow x_{\pm} = \pm \sqrt{\frac{1-f}{3a}}.$$ 

Since $G(-\infty) = -\infty, G(0) = -F,$ and $G(+\infty) = +\infty,$ by using the above relations, it implies that there exists unique positive equilibrium point $\bar{x}.$

Next result uses standard local stability analysis, see [12] and [13]. Let

$$p = \frac{\partial f(u,v)}{\partial u}(\bar{x},\bar{x}) \quad \text{and} \quad q = \frac{\partial f(u,v)}{\partial v}(\bar{x},\bar{x})$$

denote the partial derivatives of $f(u,v)$ evaluated at the equilibrium $\bar{x}$ of Equation (1.4). Then the equation

$$y_{n+1} = py_n + qy_{n-1}, \quad n = 0, 1, \ldots \quad (2.2)$$

is called the linearized equation associated with Equation (1.4) about the equilibrium point $\bar{x}.$

**Proposition 2.1.** (a) If both roots of the quadratic equation

$$\lambda^2 - p\lambda - q = 0 \quad (2.3)$$

lie in the open unit disk $|\lambda| < 1,$ then the equilibrium $\bar{x}$ of Equation (1.4) is locally asymptotically stable.

(b) If at least one of the roots of Equation (2.3) has absolute value greater than one, then the equilibrium $\bar{x}$ of Equation (1.4) is unstable.

(c) A necessary and sufficient condition for both roots of Equation (2.3) to lie in the open unit disk $|\lambda| < 1,$ is

$$|p| < 1 - q < 2. \quad (2.4)$$
In this case the locally asymptotically stable equilibrium \( \bar{x} \) is also called a sink.

(d) A necessary and sufficient condition for both roots of Equation (2.3) to have absolute value greater than one is
\[
|q| > 1 \quad \text{and} \quad |p| < |1 - q|.
\]
In this case \( \bar{x} \) is a repeller.

(e) A necessary and sufficient condition for one root of Equation (2.3) to have absolute value greater than one and for the other to have absolute value less than one is
\[
p^2 + 4q > 0 \quad \text{and} \quad |p| > |1 - q|.
\]
In this case the unstable equilibrium \( \bar{x} \) is called a saddle point.

(f) A necessary and sufficient condition for a root of Equation (2.3) to have absolute value equal to one is
\[
|p| = |1 - q| \quad \text{or} \quad (q = -1 \text{ i } |p| \leq 2).
\]
In this case the equilibrium \( \bar{x} \) is called a nonhyperbolic point.

Now, we prove the following lemma.

**Lemma 2.1.**

1. If \( f > 1 \), then the unique equilibrium point \( \bar{x} \) of Equation (1.1) is:
   i) locally asymptotically stable if \( 2(f - 1)\sqrt{a(f - 1)} - aF > 0 \),
   ii) a nonhyperbolic point if \( 2(f - 1)\sqrt{a(f - 1)} - aF = 0 \),
   iii) a saddle point if \( 2(f - 1)\sqrt{a(f - 1)} - aF < 0 \).

2. If \( f \leq 1 \), then the unique equilibrium point \( \bar{x} \) of Equation (1.1) is a saddle point.

**Proof.** Denote as
\[
H(u, v) = \frac{v + F}{au^2 + f}.
\]
Then we have
\[
p = H'_u(\bar{x}) = \frac{-2a\bar{x}(\bar{x} + F)}{(a\bar{x}^2 + f)^2}, \quad q = -H'_v(\bar{x}) = \frac{-1}{a\bar{x}^2 + f} < 0,
\]
and
\[
p - 1 - q = \frac{-2a\bar{x}(\bar{x} + F)}{(a\bar{x}^2 + f)^2} - 1 + \frac{1}{a\bar{x}^2 + f} = \frac{-3a\bar{x}^2 + 1 - f}{a\bar{x}^2 + f} = -\frac{G'(\bar{x})}{a\bar{x}^2 + f},
\]
\[
p + 1 + q = \frac{-a\bar{x}^2 + f - 1}{a\bar{x}^2 + f}.
\]
Since the function \( G(x) \) is increasing when it passes through the equilibrium point \( \bar{x} \), that is \( G'(x) > 0 \), so it implies \( p - 1 - q < 0 \). Hence, we need to determine the sign of the term \( p + 1 + q \). Since the denominator is obviously positive, the sign of the expression depends on the sign of the numerator.
\(-ax^2 + f - 1 = 0 \Rightarrow x_{\pm} = \pm \sqrt{\frac{f-1}{a}}.\)

1. Let \(f > 1.\) Since
\[2a(f-1)\sqrt{a(f-1)} - a^2F > 0 \iff G(x_+) > 0 \iff \bar{x} < x_+ \Rightarrow p + 1 + q > 0,\]
the unique equilibrium point \(\bar{x}\) is locally asymptotically stable. Analogously,
\[2a(f-1)\sqrt{a(f-1)} - a^2F = 0 \iff G(x_+) = 0 \iff \bar{x} = x_+ \Rightarrow p + 1 + q = 0,\]
which implies that the equilibrium point is nonhyperbolic. Finally, if
\[2a(f-1)\sqrt{a(f-1)} - a^2F < 0 \iff G(x_+) < 0 \iff \bar{x} > x_+ \Rightarrow p + 1 + q < 0,\]
which implies that the equilibrium point is a saddle point.

2. If \(f \leq 1,\) then \(p + q + 1 = \frac{-a\phi^2 + f - 1}{\bar{x} + f} < 0\) and the statement is true. \(\square\)

3. PERIOD-TWO SOLUTIONS

Now we present results about existence and local stability of minimal period-two solutions of Equation (1.1).

**Theorem 3.1.** Assume that \(f > 1.\) If \(aF^2 - 4(f-1)^3 > 0,\) then Equation (1.1) has a minimal period-two solution
\[\{\phi, \psi, \phi, \psi, \ldots\}\] and \[\{\psi, \phi, \psi, \phi, \ldots\}\] (3.1)
where
\[\phi = \frac{aF - \sqrt{a^2F^2 - 4a(f-1)^3}}{2a(f-1)}, \quad \psi = \frac{aF + \sqrt{a^2F^2 - 4a(f-1)^3}}{2a(f-1)},\] (3.2)
which is locally asymptotically stable.

**Proof.** Suppose that there exists a minimal period-two solution \(\{\phi, \psi, \phi, \psi, \ldots\}\) of Equation (1.1), where \(\phi\) and \(\psi\) are distinct nonnegative real numbers such that \(\phi^2 + \psi^2 \neq 0.\) Then we have the following system:
\[\begin{align*}
\phi &= \frac{\phi + F}{a\psi + f}, \\
\psi &= \frac{\psi + F}{a\phi + f},
\end{align*}\] (3.3)
which is equivalent to
\[(\phi - \psi)(f - 1 - a\phi\psi) = 0.\]
Since \(\phi \neq \psi,\) we have that
\[\phi\psi = \frac{f - 1}{a} \Rightarrow \phi = \frac{f - 1}{a\psi}, \quad f > 1.\] (3.4)
Substituting (3.4) in (3.3) we obtain
\[a \left( \frac{f - 1}{a\psi} \right)^2 \psi + f\psi = \psi + F,\]
are given by (3.2). By substitution

from which

The map

where

is of the form

\[ T(u, v) = \left( \begin{array}{c} \phi(u, v) \\ \psi(u, v) \end{array} \right) \]

The Jacobian matrix of the map

\[ J_T (\phi, \psi) = \left( \begin{array}{cc} \frac{\partial G}{\partial u} (\phi, \psi) & \frac{\partial G}{\partial v} (\phi, \psi) \\ \frac{\partial H}{\partial u} (\phi, \psi) & \frac{\partial H}{\partial v} (\phi, \psi) \end{array} \right) \]

where

and

and

\[ \frac{\partial H}{\partial v} (\phi, \psi) = \frac{ah^2(\phi, \psi) + f}{(ah^2 + f)^2} - \left( \psi + F \right) 2ah(\phi, \psi) \frac{\partial h}{\partial v} (\phi, \psi) \]

\[ = \frac{ah^2(\phi, \psi) + f}{(ah^2 + f)^2} - \left( \psi + F \right) 2ah(\phi, \psi) \frac{\partial h}{\partial v} (\phi, \psi) \]
\[
\frac{1}{(ah^2 (\phi, \psi) + f)^2} + \frac{(\Psi + F)2a^{\phi + F}2a^{\psi F}}{a^{\psi F} f (a^{\phi^2 + f})^2} \]
\[
= \frac{1}{a\phi^2 + f} + \frac{4a^2 \phi^2 \psi^2}{(a\psi^2 + f)(a\phi^2 + f)}
\]

Now we have
\[
Tr_{J_2} = p = \frac{\partial G}{\partial t} (\Phi, \Psi) + \frac{\partial H}{\partial t} (\Phi, \Psi) = \frac{1}{a\psi^2 + f} + \frac{1}{a\phi^2 + f} + \frac{4a^2 \phi^2 \psi^2}{(a\psi^2 + f)(a\phi^2 + f)},
\]
and
\[
Det_{J_2} = q = \frac{1}{a\psi^2 + f} \left( \frac{1}{a\phi^2 + f} + \frac{4a^2 \phi^2 \psi^2}{(a\psi^2 + f)(a\phi^2 + f)} \right) - \frac{2a\phi \psi}{a\psi^2 + f} \left( \frac{2a\phi \psi}{(a\psi^2 + f)(a\phi^2 + f)} \right).
\]
Notice that \( p > 0 \) and \( q < 1 \), so we just need to show that \( p < 1 + q \).

\[
p < 1 + q \iff \frac{1}{a\phi^2 + f} + \frac{1}{a\psi^2 + f} + \frac{4a^2 \phi^2 \psi^2}{(a\psi^2 + f)(a\phi^2 + f)} < 1 + \frac{1}{(a\phi^2 + f)(a\psi^2 + f)} \]
\[
\iff a\phi^2 + f + a\psi^2 + f + 4a^2 \phi^2 \psi^2 < \left( a\phi^2 + f \right) \left( a\psi^2 + f \right) + 1 \]
\[
\iff a\phi^2 (1 - f) + a\psi^2 (1 - f) - (1 - f)^2 + 3a^2 \phi^2 \psi^2 < 0 \]
\[
\iff a(1 - f) \left( \phi^2 + \psi^2 \right) - (1 - f)^2 + 3a^2 \left( \frac{f - 1}{a} \right)^2 < 0 \]
\[
\iff (1 - f) \frac{4a^2 F^2 - 8a(f - 1)^3}{4a(f - 1)^2} + 2(f - 1)^2 < 0 \]
\[
\iff -\frac{a^2 F^2 + 2a(f - 1)^3}{a(f - 1)^2} + 2(f - 1)^2 < 0 \]
\[
\iff -\frac{a^2 F^2 + 2a(f - 1)^3}{a(f - 1)^2} < 0,
\]
which is true since \( f > 1 \) and \( D = a^2 F - 4a(f - 1)^3 > 0 \). \( \square \)

4. **Global Dynamics**

In this section, we present global dynamic results for Equation (1.1). Every solution of Equation (1.1) satisfies
\[
x_{n+1} \leq \frac{x_n}{f} + \frac{F}{f}, \quad n = 0, 1, \ldots
\]
which in view of Theorem 1.4, means that \( x_n \leq z_n, \quad n = 0, 1, \ldots \), where \( \{z_n\} \) satisfies
\[
z_{n+1} = \frac{z_n}{f} + \frac{F}{f}. \quad (4.1)
\]
So we obtain that \( x_n \leq \frac{F}{f - 1} \) if \( f > 1 \) since
\[
z_n = \frac{F}{f - 1} + \frac{C_1}{\sqrt{f^n}} + \frac{(-1)^n C_2}{\sqrt{f^n}}, \quad n = 0, 1, \ldots
\]
So, every solution of Equation (1.1) is bounded if \( f > 1 \) and in that case \([L, U] = [0, \frac{F}{f-1}]\) is an invariant interval for solutions of the Equation (1.1).

**Theorem 4.1.** If \( f > 1 \) and \( D = a^2 F - 4a(f-1)^3 > 0 \) then there exist equilibrium point \( \bar{x} \) which is a saddle point and the minimal period-two solution defined by (3.1) and (3.2) which is locally asymptotically stable. There exists a set \( C \subset \mathbb{R} = (0, \infty) \times (0, \infty) \) such that \((x_+, \bar{x}_+) \in C, \) and \( W^s((x_+, \bar{x}_+)) = C \) is an invariant subset of the basin of attraction of \((x_+, \bar{x}_+)\). The set \( C \) is a graph of a strictly increasing continuous function of the first variable on an interval and separates \( \mathbb{R} \) into two connected and invariant components \( W_-(x_+, \bar{x}_+) \) and \( W_+(x_+, \bar{x}_+) \), which satisfy that

(i) if \((x_-, x_0) \in W_+(x_+, \bar{x}_+)\), then

\[
\lim_{n \to \infty} x_{2n} = \frac{aF - \sqrt{a^2 F^2 - 4a(f-1)^3}}{2a(f-1)}
\]

and

\[
\lim_{n \to \infty} x_{2n+1} = \frac{aF + \sqrt{a^2 F^2 - 4a(f-1)^3}}{2a(f-1)};
\]

(ii) if \((x_-, x_0) \in W_-(x_+, \bar{x}_+)\), then

\[
\lim_{n \to \infty} x_{2n} = \frac{aF + \sqrt{a^2 F^2 - 4a(f-1)^3}}{2a(f-1)}
\]

and

\[
\lim_{n \to \infty} x_{2n+1} = \frac{aF - \sqrt{a^2 F^2 - 4a(f-1)^3}}{2a(f-1)}.
\]

For visual representation see Figure 1.

**Figure 1.** Global dynamics of Equation (1.1) for \( f = 5, F = 12, a = 2 \) and initial conditions \((x_0, x_{-1}) = (0.3, 0.1) - \text{red} \) and \((x_0, x_{-1}) = (1.6, 1.7) - \text{green} \).
Proof. It is clear that the point \((\overline{x}, \overline{x})\) and the period-two solutions \((\phi, \psi)\) and \((\psi, \phi)\) are the equilibrium points of the map \(T^2\). Since the map \(T^2\) is competitive, by Theorems 1.2 and 1.3, there exists a curve \(C\) through \((\overline{x}, \overline{x})\) that is invariant and a subset of the basin of attraction of \((\overline{x}, \overline{x})\) and \(C\) is the graph of a strictly increasing continuous function of the first coordinate on an interval. If \((u_0, v_0) \in W_+((\overline{x}, \overline{x}))\), then by Theorem 1.3, \(T^{2n}(u_0, v_0) \in W_+((\overline{x}, \overline{x}))\) and \(T^{2n+1}(u_0, v_0) \in W_-(\overline{x}, \overline{x})\) for all \(n \in \{0, 1, 2, \ldots\}\). So we obtain that

\[
\lim_{n \to \infty} T^{2n}(u_0, v_0) = (\psi, \phi) \quad \text{and} \quad \lim_{n \to \infty} T^{2n+1}(u_0, v_0) = (\phi, \psi).
\]

If \((u_0, v_0) \in W_-((\overline{x}, \overline{x}))\), then \(T^{2n}(u_0, v_0) \in W_-((\overline{x}, \overline{x}))\) and \(T^{2n+1}(u_0, v_0) \in W_+((\overline{x}, \overline{x}))\) for all \(n \in \{0, 1, 2, \ldots\}\) which yields

\[
\lim_{n \to \infty} T^{2n}(u_0, v_0) = (\phi, \psi) \quad \text{and} \quad \lim_{n \to \infty} T^{2n+1}(u_0, v_0) = (\psi, \phi).
\]

Consequently, if \((x_{-1}, x_0) \in W_+((\overline{x}, \overline{x}))\), then

\[
\lim_{n \to \infty} T^{2n}(x_{-1}, x_0) = (\psi, \phi) \quad \text{and} \quad \lim_{n \to \infty} T^{2n+1}(x_{-1}, x_0) = (\phi, \psi),
\]

which means that \(\lim_{n \to \infty} x_{2n} = \phi\) and \(\lim_{n \to \infty} x_{2n+1} = \psi\). If \((x_{-1}, x_0) \in W_-((\overline{x}, \overline{x}))\), then

\[
\lim_{n \to \infty} T^{2n}(x_{-1}, x_0) = (\psi, \phi) \quad \text{and} \quad \lim_{n \to \infty} T^{2n+1}(x_{-1}, x_0) = (\phi, \psi),
\]

which means that \(\lim_{n \to \infty} x_{2n} = \psi\) and \(\lim_{n \to \infty} x_{2n+1} = \phi\), where

\[
\phi = \frac{aF - \sqrt{a^2F^2 - 4a(f-1)^3}}{2a(f-1)} \quad \text{and} \quad \psi = \frac{aF + \sqrt{a^2F^2 - 4a(f-1)^3}}{2a(f-1)}.
\]

\[\square\]

Theorem 4.2. If \(f > 1\) and \(D \leq 0\), then the unique equilibrium solution of Equation (1.1) is globally asymptotically stable.

Proof. The proof follows from Theorems 1.1 and 3.1 and Lemma 2.1. \[\square\]

Theorem 4.3. If \(f \leq 1\), then Equation (1.1) has a unique equilibrium point \(\overline{x}\) which is a saddle point and has no minimal period-two solutions. There exists a set \(C\) which is an invariant subset of the basin of attraction of \((\overline{x}, \overline{x})\). The set \(C\) is a graph of a strictly increasing continuous function of the first variable on an interval and separates \(R\) into two connected and invariant components \(W_-(\overline{x}+, \overline{x}+)\) and \(W_+(\overline{x}+, \overline{x}+)\), which satisfy that

(i) (i) if \((x_{-1}, x_0) \in W_-(\overline{x}+, \overline{x}+)\), then

\[
\lim_{n \to \infty} x_{2n} = \infty \quad \text{and} \quad \lim_{n \to \infty} x_{2n+1} = 0;
\]

(ii) (ii) if \((x_{-1}, x_0) \in W_+(\overline{x}+, \overline{x}+)\), then

\[
\lim_{n \to \infty} x_{2n} = 0 \quad \text{and} \quad \lim_{n \to \infty} x_{2n+1} = \infty.
\]
See Figure 2. for visual representation.

**Figure 2.** Global dynamics of Equation (1.1) for \(f = 1, F = 0.15, a = 3\) and initial conditions \((x_0, x_{-1}) = (0.3, 0.1) - \text{red}\) and \((x_0, x_{-1}) = (0.7, 0.9) - \text{green}\).

**Proof.** The point \((\bar{x}, \bar{y})\) is a saddle point for the strictly competitive map \(T^2\) as well. The existence of the set \(C\) with the stated properties follows from Lemma 2.1, Theorems 1.2, 1.3 and 3.1. Equation (1.1) is equivalent to the system of difference equations (3.5), which can be decomposed into the system of the even-indexed and odd-indexed terms as follows:

\[
\begin{align*}
\begin{cases}
  u_{2n} &= v_{2n-1}, \\
  u_{2n+1} &= v_{2n}, \\
  v_{2n} &= \frac{u_{2n-1} + F}{av_{2n-1}^2 + f}, \\
  v_{2n+1} &= \frac{u_{2n} + F}{av_{2n}^2 + f}.
\end{cases}
\end{align*}
\]  

(4.2)

Now, using (4.2) we obtain

i) if \((u_0, v_0) \in \mathcal{W}_-\), then

\[(u_{2n}, v_{2n}) = T^{2n}((u_0, v_0)) \rightarrow (0, \infty)\]

and

\[(u_{2n+1}, v_{2n+1}) = T^{2n+1}((u_0, v_0)) \rightarrow (\infty, 0);\]

ii) if \((u_0, v_0) \in \mathcal{W}_+\), then

\[(u_{2n}, v_{2n}) = T^{2n}((u_0, v_0)) \rightarrow (\infty, 0)\]

and

\[(u_{2n+1}, v_{2n+1}) = T^{2n+1}((u_0, v_0)) \rightarrow (0, \infty);\]

Consequently,
i) if $(x_{-1}, x_0) \in W^-(\overline{\{x, \overline{x}\}})$, then
$$T^{2n}(x_{-1}, x_0) \to (0, \infty) \quad \text{and} \quad T^{2n+1}(x_{-1}, x_0) \to (\infty, 0),$$
that is
$$\lim_{n \to \infty} x_{2n} = \infty \quad \text{and} \quad \lim_{n \to \infty} x_{2n+1} = 0.
$$

ii) if $(x_{-1}, x_0) \in W^+(\overline{\{x, \overline{x}\}})$, then
$$T^{2n+1}(x_{-1}, x_0) \to (0, \infty) \quad \text{and} \quad T^{2n}(x_{-1}, x_0) \to (\infty, 0),$$
that is
$$\lim_{n \to \infty} x_{2n} = 0 \quad \text{and} \quad \lim_{n \to \infty} x_{2n+1} = \infty. \quad \square$$

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REFERENCES


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CESÀRO MEANS OF SUBSEQUENCES OF DOUBLE SEQUENCES

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In memory of Professor Harry I. Miller

ABSTRACT. In this paper we characterize the convergence and \((C, 1, 1)\) summa-
bility of a double sequence. In particular we study conditions under which the
convergence or \((C, 1, 1)\) summability of a double sequence carry over to that
of its subsequences, and conversely, whether these properties for suitable sub-
sequences imply them for the sequence itself. We show, for instance, that a
bounded double sequence is \((C, 1, 1)\) summable if and only if almost all of its
subsequences are \((C, 1, 1)\) summable.

1. INTRODUCTION

Establishing a one-to-one correspondence between the interval \((0, 1]\) and the col-
lection of all subsequences of a given sequence \((s_n)\), Buck and Pollard [2] proved
that \((s_n)\) is \((C, 1)\) summable if almost all of subsequences are, but not conversely.
Replacing \((C, 1)\) matrix by \(p\)-Cesàro matrix similar problems have also been con-
sidered in [9].

In the present paper we consider analogous problems for double sequences.
A double sequence \(s = (s_{ij})\) is said to be Pringsheim convergent (i.e., it is con-
vergent in Pringsheim’s sense) to \(L\) if for every \(\varepsilon > 0\) there exists an \(N \in \mathbb{N}\) such
that \(|s_{ij} - L| < \varepsilon\) whenever \(i, j \geq N\) ( [10]). In this case \(L\) is called the Pringsheim
limit of \(s\) and the space of such sequences is denoted by \(c^{(2)}\). A double sequence \(s\)
is bounded if there exists a positive number \(M\) such that \(|s_{ij}| < M\) for all \(i\) and \(j\),
i.e.,

\[
\|s\|_{(\infty, 2)} = \sup_{i,j} |s_{ij}| < \infty.
\]

We will denote the set of all bounded double sequences by \(l^{(2)}_{\infty}\). Note that in contrast
to the case for single sequences, a convergent double sequence need not to be
bounded.

Throughout the paper convergence means the Pringsheim convergence.

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double subsequences.
Four dimensional Cesàro matrix $(C, 1, 1) = \left( c_{nm}^{jk} \right)$ is defined by

$$c_{nm}^{jk} = \begin{cases} \frac{1}{nm}, & 1 \leq j \leq n \text{ and } 1 \leq k \leq m \\ 0, & \text{otherwise} \end{cases}.$$ 

It is known that the $(C, 1, 1)$ matrix is an RH regular, i.e., it sums every bounded convergent sequence to the same limit.

There exist several versions of the concept of subsequences for double sequences ([5], [11], [16]). We adopt the definition of [5, 6] on subsequences of double sequences throughout the paper.

Let $X$ denote the set of all double sequences of 0’s and 1’s, that is

$$X = \{ x = (x_{jk}) : x_{jk} \in \{0, 1\} \text{ for each } j, k \in \mathbb{N} \}.$$ 

Let $\mathcal{R}$ be the smallest $\sigma$-algebra of subsets of the set $X$ which contains all sets of the form

$$\{ x = (x_{jk}) \in X : x_{j_1k_1} = a_1, \ldots, x_{j_nk_n} = a_n \}$$

where each $a_i \in \{0, 1\}$ and the pairs $\{(j_i k_i)\}_{i=1}^n$ are pairwise distinct.

There exists a unique probability measure $P$ on the set $\mathcal{R}$, such that

$$P\left( \{ x = (x_{jk}) \in X : x_{j_1k_1} = a_1, \ldots, x_{j_nk_n} = a_n \} \right) = \frac{1}{2^n}$$

for all choices of $n$ and all pairwise disjoint pairs $\{(j_i k_i)\}_{i=1}^n$, and all choices of $a_1, \ldots, a_n$ ([5]).

Let $s = (s_{jk})$ be a double sequence and $x = (x_{jk}) \in X$. Following [5] we define a subsequence of the sequence $s$ by

$$s_{jk}(x) = \begin{cases} s_{jk} & \text{if } x_{jk} = 1 \\ * & \text{if } x_{jk} = 0 \end{cases}.$$ 

Mapping $x \to s(x)$ is a bijection from the set $X$ to the set of all the subsequences of the sequence $s = (s_{jk})$.

An element $x$ of $X$ is said to be normal ([5]) if for each $\varepsilon > 0$ there is a natural number $N_\varepsilon$ such that for $n, m \geq N_\varepsilon$ we have $\left| \frac{1}{nm} \sum_{j \leq n, k \leq m} x_{jk} - \frac{1}{2} \right| < \varepsilon$. Let $\eta$ denote the set of all elements $x$ in $X$ that are normal. This means that normal elements are $(C, 1, 1)$-summable to $\frac{1}{2}$. It is also known ([5]) that $P(\eta) = 1$. We also need the functions $r_{j,k}(x) = 2x_{j,k} - 1$, for $x = (x_{jk}) \in X$. Recall that the functions $r_{j,k}$ are the Rademacher functions (see [5]).

2. Subsequence Characterization of Convergence and Cesàro Summability

In this section we characterize the convergence and $(C, 1, 1)$ summability of a double sequence. In particular we study conditions under which the convergence
or \((C, 1, 1)\) summability of a double sequence carry over to that of its subsequences, and conversely, whether these properties for suitable subsequences imply them for the sequence itself. The results are analogous to those of Buck and Pollard [2] for single sequences. We note in passing that the summability properties of the set of second category subsequences may be found in [12].

**Theorem 2.1.** If almost all of the subsequences of a double sequence \(s = (s_{jk})\) converges to \(L\), then the sequence \(s = (s_{jk})\) itself converges to \(L\).

**Proof.** Assume that almost all of the subsequences of a double sequence \(s = (s_{jk})\) converges to \(L\), i.e., \(P(C) = 1\) where \(C = \{x \in X : s(x)\) converges to \(L\}\).

We use the technique given in [5]. Now given a sequence \(x = (x_{jk}) \in X\) we define a sequence \(\bar{x} = (\bar{x}_{jk})\) by

\[
\bar{x}_{jk} = \begin{cases} 
0, & x_{jk} = 1 \\
1, & x_{jk} = 0 
\end{cases}
\]

Let \(Y = C \cap \eta\) and \(\bar{Y} = \{(x_{jk}) : x_{jk} \in Y\}\). Therefore we have \(\bar{Y} = \bar{C} \cap \eta\) where \(\bar{C}\) is defined in the obvious way. Since the mapping \((x_{jk}) \rightarrow (\bar{x}_{jk})\) preserves the measure \(P\), we get \(P(\bar{Y}) = 1\) and hence \(P(Y \cap \bar{Y}) = 1\). So \(Y \cap \bar{Y}\) is a non-empty set. If \(x = (x_{jk}) \in Y \cap \bar{Y}\), then we have \(x \in C\), \(x \in \eta\) and \(\bar{x} \in C\), \(\bar{x} \in \eta\). Since \(x, \bar{x} \in C\), we have \(s(x) \rightarrow L\) and \(s(\bar{x}) \rightarrow L\) with \(x, \bar{x} \in \eta\). This implies that the sequence \(s = (s_{jk})\) converges to \(L\). \(\Box\)

We now turn our attention to the \((C, 1, 1)\)-summability of subsequences.

**Theorem 2.2.** If almost all subsequences of \(s = (s_{jk})\) are \((C, 1, 1)\)-summable to a value \(L\) then the sequence \(s = (s_{jk})\) is \((C, 1, 1)\)-summable to \(L\).

**Proof.** If almost all subsequences of \((s_{jk})\) are \((C, 1, 1)\)-summable to a value \(L\) then the set \(G = \{x \in X : s(x)\) is \((C, 1, 1)\)-summable to \(L\}\) has probability measure 1. Using the same type of argument in Theorem 2.1, if \(x \in G \cap \eta\) then we get \(\bar{x} \in G \cap \eta\). Hence we obtain

\[s(x) \rightarrow L(C, 1, 1)\]

and

\[s(\bar{x}) \rightarrow L(C, 1, 1),\]

with \(x, \bar{x} \in \eta\). That is

\[
\lim_{n,m \rightarrow \infty} \frac{\sum_{j,k=1}^{n,m} s_{jk} x_{jk}}{\sum_{j,k=1}^{n,m} x_{jk}} = L
\]

and similarly we get
Also since \( x, \overline{x} \in \eta \), we have
\[
\lim_{n,m \to \infty} \frac{1}{nm} \sum_{j,k=1}^{n,m} x_{jk} = \frac{1}{2} \quad \text{and} \quad \lim_{n,m \to \infty} \frac{1}{nm} \sum_{j,k=1}^{n,m} \overline{x}_{jk} = \frac{1}{2}.
\]

On the other hand the \((C,1,1)\)-summability of the sequence \((s_{jk})\) is equivalent to the existence of the limit of the following expression
\[
\frac{\sum_{j,k=1}^{n,m} s_{jk}}{nm} \sum_{j,k=1}^{n,m} x_{jk} + \frac{\sum_{j,k=1}^{n,m} \overline{s}_{jk} x_{jk}}{nm} \sum_{j,k=1}^{n,m} \overline{x}_{jk},
\]
so we get
\[
\lim_{n,m \to \infty} \frac{\sum_{j,k=1}^{n,m} s_{jk}}{nm} = \frac{L}{2} + \frac{L}{2} = L,
\]
which means that the sequence \((s_{jk})\) is \((C,1,1)\)-summable to \(L\).

In order to get the converse of Theorem 2.2, we need the following lemmas. The first one is an analog of the Khintchine inequality \([3]\).

**Lemma 2.3.** Let \( t_{nm}(x) = \sum_{j,k=1}^{n,m} s_{jk}^{r_{jk}}(x) \), \( B_{nm} = \sum_{j,k=1}^{n,m} s_{jk}^{2} \). Then the following inequality
\[
E \left( (t_{nm})^{2r} \right) \leq \frac{(2r)!}{2^{2r!}} (B_{nm})^{r}
\]
is fulfilled, where \( r \) is a positive integer.

**Proof.** \[
E \left( (t_{nm})^{2r} \right) = \sum_{v_1 + \ldots + v_i = 2r} A_{v_1,\ldots,v_i} \sum_{j,k=1}^{n,m} v_{1j}^{v_{1j}^{v_{1j}}} \ldots v_{ij}^{v_{ij}^{v_{ij}}} E \left[ r_{j_{1k_{1}}^{v_{1j}}} \ldots r_{j_{ik_{i}}^{v_{ij}}} (x) \right]
\]
and \( 1 \leq j_1, \ldots, j_i \leq n, 1 \leq k_1, \ldots, k_i \leq m \) where \( \sum_{\mu=1}^{i} v_{\mu} = 2r, A_{v_1,\ldots,v_i} = \frac{(v_1 + \ldots + v_i)!}{v_1! \ldots v_i!} \).

We have
\[
E \left[ r_{j_{1k_{1}}}^{v_{1}} (x) \ldots r_{j_{ik_{i}}}^{v_{i}} (x) \right] = \left\{ \begin{array}{ll} 1 & , v_1, \ldots, v_i \text{ even} \\
0 & , \text{otherwise} \end{array} \right.
\]
and hence
\[
E \left( (t_{nm})^{2r} \right) = \sum_{p_1 + \ldots + p_i = r} A_{2p_1,\ldots,2p_i} \sum_{j_{1k_{1}}}^{2p_{1}} \ldots \sum_{j_{ik_{i}}}^{2p_{i}}
\]
where \( \sum_{u=1}^{i} p_u = r \) such that \( p_1, \ldots, p_i \) are positive integers. On the other hand it is well known that
\[
(2p_1)! \ldots (2p_i)! \geq 2^{p_1} p_1! \ldots 2^{p_i} p_i!.
\]
So we have
\[
E \left( (t_{nm})^{2r} \right) = \sum_{p_1 + \ldots + p_r = r} \frac{(2r)!}{2^{2r} r!} \frac{r!}{p_1! \ldots p_r!} \sum_{j} s_{j_1} \ldots s_{j_k}^{2p_i} \\
\leq \frac{(2r)!}{2^{2r} r!} \sum_{p_1 + \ldots + p_r = r} \frac{r!}{p_1! \ldots p_r!} \sum_{j} s_{j_1} \ldots s_{j_k}^{2p_i} \\
= \frac{(2r)!}{2^{2r} r!} (B_{nm})^r.
\]

This completes the proof. \(\square\)

The next result is an analog of the Marcinkiewicz-Zygmund inequality [15].

**Lemma 2.4.** Let \(t_{nm}(x) = \sum_{j,k=1}^{n,m} s_{jk} r_{jk}(x)\), \(B_{nm} = \sum_{j,k=1}^{n,m} s_{jk}^2\) and \(t_{nm}^+(x) = \max_{1 \leq j \leq n, 1 \leq k \leq m} |t_{jk}|\). Then for \(a > 0\) the following inequality

\[
E \left( e^{at_{nm}^+(x)} \right) \leq 32 e^{a^2 B_{nm} / 2}
\]

holds.

**Proof.** Observe that

\[
e^{at_{nm}^+(x)} \leq 2 \frac{e^{at_{nm}^+(x)} + e^{-at_{nm}^+(x)} }{2} \\
= 2 \left\{ \sum_{r=0}^{\infty} \frac{(at_{nm}^+(x))^r}{r!} + \sum_{r=0}^{\infty} \frac{(-1)^r(at_{nm}^+(x))^r}{r!} \right\} \\
= 2 \left\{ 1 + \sum_{r=1}^{\infty} \frac{(at_{nm}^+(x))^{2r}}{(2r)!} \right\}.
\]

Using the last inequality we get

\[
E \left( e^{at_{nm}^+(x)} \right) \leq 2 \left\{ 1 + \sum_{r=1}^{\infty} \frac{E \left[ (at_{nm}^+(x))^{2r} \right]}{(2r)!} \right\} \\
\leq 2 \left\{ 1 + \sum_{r=1}^{\infty} \frac{1}{(2r)!} a^{2r} E \left[ \left( \max_{1 \leq j \leq n, 1 \leq k \leq m} |t_{jk}| \right)^{2r} \right] \right\}.
\]

Since \(E \left( r_{jk}(x) \right) = 0\), \(X_{jk} := s_{jk} r_{jk}\) for \(j \geq 1, k \geq 1\) is an array of martingale differences (see [13]). Hence using the Doob inequality [4] for multiple sequences and considering Lemma 2.3, we get
Theorem 2.5. If the sequence \((s_{jk})\) is \((C, 1, 1)\)-summable to a value \(L\) and
\[
\sum_{j,k=1,1}^{n,m} s_{jk}^2 = o \left( \frac{n^2 m^2}{\log \log nm} \right)
\]
then almost all subsequences of \((s_{jk})\) are \((C, 1, 1)\)-summable to \(L\).

Proof. The \((C, 1, 1)\)-summability of almost all subsequences of \((s_{jk})\) is equivalent to the convergence of the following expression
\[
\frac{\sum_{j,k=1,1}^{n,m} s_{jk} x_{jk}}{\sum_{j,k=1,1}^{n,m} x_{jk}} \quad \text{for almost all } x.
\]

We can rewrite the above expression as follows for almost all \(x\)
\[
\frac{\sum_{j,k=1,1}^{n,m} s_{jk} \left( \frac{1 + r_{jk}(x)}{2} \right)}{\sum_{j,k=1,1}^{n,m} \left( \frac{1 + r_{jk}(x)}{2} \right)} = \frac{1}{2nm} \sum_{j,k=1,1}^{n,m} s_{jk} + \frac{1}{2nm} \sum_{j,k=1,1}^{n,m} s_{jk} r_{jk}(x) = \ \frac{1}{nm} \sum_{j,k=1,1}^{n,m} \left( \frac{1 + r_{jk}(x)}{2} \right).
\]

Since \(P(\eta) = 1\), observe that the denominator of (2.1) converges to \(\frac{1}{2}\) for almost all \(x\). To complete the proof, it suffices to establish that
\[
\frac{1}{nm} \sum_{j,k=1,1}^{n,m} s_{jk} r_{jk}(x) \to 0, \text{ (as } n,m \to \infty) \quad \text{for almost all } x.
\]

Let \(\varepsilon > 0\) and define
\[
E_{jk} := \{x : \text{there exists } (n,m) \text{ with } 2j^{-1} < n \leq 2^j, 2k^{-1} < m \leq 2^k \text{ such that } |t_{nm}(x)| \geq n\varepsilon \}
\]
and let
\[ G_{jk} = \{ x : t_{2j,2k}^* (x) > 2^{j-1} 2^{k-1} \varepsilon \} . \]
Notice that \( E_{jk} \subset G_{jk} \). The proof will be completed if we prove that for every \( \varepsilon > 0 \),
\[ \sum_{j,k=1,1}^\infty P(G_{jk}) < \infty . \]
Now using Lemma 2.4 we have
\[ P(G_{jk}) e^{a 2^{j-1} 2^{k-1} \varepsilon} = \int_X e^{a t_{2j,2k}^* (x)} dP(x) \leq \mathbf{E} \left( e^{a t_{2j,2k}^* (x)} \right) \leq 32 e^{a \varepsilon}. \]

Hence
\[ P(G_{jk}) \leq 32 e^{a \varepsilon}. \]
Taking \( a = \frac{2^{j-1} 2^{k-1} \varepsilon}{B_{2j,2k}} \), we have
\[ P(G_{jk}) \leq 32 e^{\frac{\varepsilon^2 2^{2(j-1)} 2^{2(k-1)}}{2B_{2j,2k}}} \]
\[ \leq 32 e^{-\frac{\varepsilon^2}{32B_{2j,2k}}} \quad \text{(2.2)} \]

On the other hand it follows from the hypothesis that
\[ \frac{B_{2j,2k}}{(2j)^2 (2k)^2} = o \left( \frac{1}{\log \log 2^{2k}} \right) \]
\[ \leq \frac{\varepsilon^2}{96 \log \log 2^{2k}}. \]

Then (2.2) yields that
\[ P(G_{jk}) \leq 32 e^{\frac{-\varepsilon^2 96 \log \log 2^{2k} \varepsilon^2}{e^2}} \]
\[ = 32 e^{-3 \log \log 2^{2k}} \]
\[ = \frac{32}{[(j+k) \log 2]^3} . \]

Since \( \sum_{j,k=1,1}^\infty \frac{1}{[(j+k) \log 2]^3} < \infty \) (see [1]),
\[ \sum_{j,k=1,1}^\infty P(G_{jk}) \leq 32 \sum_{j,k=1,1}^\infty \frac{1}{[(j+k) \log 2]^3} < \infty . \]

Hence we obtain \( \lim_{j,k \to \infty} P(G_{jk}) = 0 \) and also \( \lim_{j,k \to \infty} P(E_{jk}) = 0 \). This completes the proof. \( \square \)

Now we are in a position to give a criterion.
Corollary 2.1. A bounded double sequence \((s_{jk})\) is \((C, 1, 1)\)-summable if and only if the almost all subsequences are \((C, 1, 1)\)-summable.

Theorem 2.6. If
\[
\lim_{n,m \to \infty} \frac{1}{nm} \sum_{j,k=1,1}^{n,m} s_{jk} r_{jk}(x) = 0 \quad \text{for almost all } x \tag{2.3}
\]
then
\[
\lim_{n,m \to \infty} \frac{1}{n^2 m^2} \sum_{j,k=1,1}^{n,m} s_{jk}^2 = 0
\]
holds.

Proof. Let \(E[p, q] = \{ (j, k) : p \leq j \leq n \text{ or } q \leq k \leq m \} \) and
\[
T_{p,q,n,m}(x) = \sum_{(j,k) \in E[p, q]} s_{jk} r_{jk}(x).
\]
Hence
\[
T_{p,q,n,m}^2(x) = \sum_{(j,k) \in E[p, q]} s_{jk}^2 + 2 \sum_{(j_1,k_1), (j_2,k_2) \in E[p, q]} s_{j_1,k_1} s_{j_2,k_2} r_{j_1,k_1}(x) r_{j_2,k_2}(x).
\]
Because of the Egoroff theorem there exists a set \(D \subset X\) with positive measure such that the limit in (2.3) exists uniformly on \(D\). Therefore
\[
\int_D T_{p,q,n,m}^2(x) \, dP(x) = P(D) \sum_{(j,k) \in E[p, q]} s_{jk}^2 + K, \tag{2.4}
\]
where
\[
K = 2 \sum_{(j_1,k_1), (j_2,k_2) \in E[p, q]} s_{j_1,k_1} s_{j_2,k_2} \int_D r_{j_1,k_1}(x) r_{j_2,k_2}(x) \, dP(x).
\]
By the Hölder inequality we have
\[
|K| \leq 2 \left( \sum_{(j_1,k_1), (j_2,k_2) \in E[p, q]} s_{j_1,k_1}^2 s_{j_2,k_2}^2 \right)^{\frac{1}{2}} \left( \sum_{(j_1,k_1), (j_2,k_2) \in E[p, q]} v_{j_1,k_1,j_2,k_2}^2 \right)^{\frac{1}{2}} \tag{2.5}
\]
where \(v_{j_1,k_1,j_2,k_2} = \int_D r_{j_1,k_1}(x) r_{j_2,k_2}(x) \, dP(x)\). We know that the functions \(r_{j_1,k_1}(x)\) and \(r_{j_2,k_2}(x)\) are orthogonal on \(X\) (see [5]). So by the Bessel inequality for double sequences we get
\[
\sum_{1 \leq j_1 < j_2 \leq \infty \atop 1 \leq k_1 < k_2 \leq \infty} v_{j_1,k_1,j_2,k_2}^2 \leq \int_X (\chi_D(x))^2 \, dP(x) = P(D).
\]
For sufficiently large \( p \) and \( q \), we have
\[
\left( \sum_{(j_1, k_1), (j_2, k_2) \in E[p, q]} v_{j_1 k_1 j_2 k_2}^2 \right)^{\frac{1}{2}} \leq \frac{P(D)}{4}.
\]

It follows from (2.5) that
\[
|K| \leq \left( \sum_{(j_1, k_1), (j_2, k_2) \in E[p, q]} s_{j_1 k_1}^2 s_{j_2 k_2}^2 \right)^{\frac{1}{2}} \frac{P(D)}{2} \leq \frac{P(D)}{2} \sum_{(j_1, k_1) \in E[p, q]} s_{j_1 k_1}^2.
\]
Combining this with (2.4) we get
\[
\int_D T_{p, q, n, m}^2(x) \, dP(x) = P(D) \sum_{(j, k) \in E[p, q]} s_{jk}^2 + K \geq \frac{P(D)}{2} \sum_{(j, k) \in E[p, q]} s_{jk}^2.
\]

By (2.3) we have that
\[
\lim_{n, m \to \infty} \frac{1}{n^2 m^2} \sum_{(j, k) \in E[p, q]} s_{jk}^2 = 0 \quad \text{and} \quad \lim_{n, m \to \infty} \frac{1}{n^2 m^2} \sum_{j, k=1}^{n, m} s_{jk}^2 = 0.
\]

This completes the proof. \( \square \)

In the next example we present a sequence so that it is \((C, 1, 1)\) summable but almost none of its subsequences are \((C, 1, 1)\) summable.

**Example 2.1.** Consider the double sequence \( s_{jk} = (-1)^j (-1)^k \sqrt{j} \sqrt{k} \). Then
\[
\sum_{j=1}^{\infty} \frac{(-1)^j \sqrt{j}}{j} = \sum_{j=1}^{\infty} \frac{(-1)^j}{\sqrt{j}} \quad \text{is convergent in the ordinary sense},
\]
and
\[
\sum_{k=1}^{\infty} \frac{(-1)^k \sqrt{k}}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} \quad \text{is convergent in the ordinary sense}.
\]

On the other hand the double series \( \sum_{j, k=1}^{\infty} \frac{(-1)^j (-1)^k}{\sqrt{j} \sqrt{k}} \) is convergent (see [1], page 90). Also since
\[
\sum_{j=1}^{\infty} \frac{(-1)^j (-1)^k}{\sqrt{j} \sqrt{k}} \quad \text{is convergent for } j = 1, 2, \ldots
\]
and
\[
\sum_{k=1}^{\infty} \frac{(-1)^j (-1)^k}{\sqrt{j} \sqrt{k}} \quad \text{is convergent for } k = 1, 2, \ldots
\]
then the double series \(\sum_{j,k=1,1}^{\infty} \frac{(-1)^j (-1)^k}{\sqrt{j\sqrt{k}}}\) is convergent in the restricted sense by Theorem 1 of [7]. Since the series \(\sum_{j,k=1,1}^{\infty} \frac{(-1)^j (-1)^k \sqrt{j\sqrt{k}}}{jk}\) is convergent in the restricted sense, we get that the sequence \(\left\{\frac{1}{nm} \sum_{j,k=1,1}^{n,m} (-1)^j (-1)^k \sqrt{j\sqrt{k}}\right\}\) converges to 0 in the Pringsheim sense [8]. Hence the sequence \(\left((-1)^j (-1)^k \sqrt{j\sqrt{k}}\right)\) is \((C,1,1)\)-summable to 0. On the other hand, since
\[
\frac{1}{n^2 m^2} \sum_{j,k=1,1}^{n,m} jk = \frac{1}{n^2 m^2} \frac{n(n+1)}{2} \frac{m(m+1)}{2} \to \frac{1}{4} \neq 0
\]
by Theorem 2.6
\[
\lim_{n,m} \frac{1}{nm} \sum_{j,k=1,1}^{n,m} (-1)^j (-1)^k \sqrt{j\sqrt{k}} r_{jk}(x) \neq 0
\]
so almost none of its subsequences are \((C,1,1)\)-summable to zero.

**References**


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QUASI-ASYMPTOTICALLY ALMOST PERIODIC VECTOR-VALUED GENERALIZED FUNCTIONS

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Dedicated to the memory of Academician Fikret Vajzović

ABSTRACT. In this paper are introduced the notions of a quasi-asymptotically almost periodic distributions and quasi-asymptotically almost periodic ultradistributions with values in a Banach space, as well as some other generalizations of these concepts. Furthermore, some applications of the introduced concepts in the analysis of systems of ordinary differential equations are provided.

1. INTRODUCTION AND PRELIMINARIES

The main goal of this paper is the selection and structural analysis of various classes of almost and asymptotically almost periodic or automorphic distributions and ultradistributions.

The concept of almost periodicity was introduced in [3] and later this theory is generalized by many other mathematicians. We put \( I = \mathbb{R} \) or \( I = [0, \infty) \), and \( f: I \to X \) be continuous function. Given \( \varepsilon > 0 \), we call \( \tau > 0 \) an \( \varepsilon \)-period for \( f(\cdot) \) iff

\[
\|f(t + \tau) - f(t)\| \leq \varepsilon, \quad t \in I.
\]

The set constituted of all \( \varepsilon \)-periods for \( f(\cdot) \) is denoted by \( \vartheta(f, \varepsilon) \). It is said that \( f(\cdot) \) is almost periodic, (AP) for short, iff for each \( \varepsilon > 0 \) the set \( \vartheta(f, \varepsilon) \) is relatively dense in \( I \), which means that there exists \( l > 0 \) such that any subinterval of \( I \) of length \( l \) meets \( \vartheta(f, \varepsilon) \). The vector space consisting of all almost periodic functions is denoted by \( \text{AP}(I; X) \).

The notion of a scalar-valued asymptotically almost periodic ((AAP) in short) distribution has been introduced by I. Cioranescu in [12], while the notion of a vector-valued (AAP) distributions has been considered by D. N. Cheban [9] following a different approach (see also I. K. Dontvi [14] and A. Halanay, D. Wexler [15]). Some contributions have been also given by B. Stanković [29]- [30].

In our recent joint research study [26], we have analyzed the notions of an almost automorphic ((AAut) in short) distributions and an almost automorphic

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(AAut) ultradistributions in Banach space; the notion of an (AP) ultradistribution in Banach space has been recently analyzed by M. Kostić [24] within the framework of Komatsu’s theory of ultradistributions, with the corresponding sequences not satisfying the condition (M.3); see also the papers by I. Cioranescu [11] and M. C. Gómez-Collado [17] for first results in this direction. As mentioned in the abstract, the main aim of this paper is to introduce the notions of a quasi-asymptotically almost periodic ((Q-AP) in short) (ultra)distribution in Banach space, as well as to provide some applications in the qualitative analysis of vector-valued (ultra)distributional solutions to systems of ordinary differential equations (the notion of a (Q-AP) ultradistribution seems to be not considered elsewhere even in scalar-valued case). In such a way, we expand and contemplate the results obtained in [5]- [6], [9], [10]- [12], [14]- [15] and [29]- [30]; see also [33]- [34] for some other results about the existence and uniqueness of various types of generalized (AP) solutions of nonlinear Volterra integro-differential equations.

The organization of paper is briefly described as follows. After giving some preliminary results and definitions from the theory of vector-valued ultradistributions (Subsection 1.1), in Section 2 we analyze the notions of (Q-AP) and Stepanov \(p\)-quasi-asymptotically almost periodic ((SpQ-AAP) in short) vector-valued (ultra)distributions. Here, we recognize the importance of condition \(T \ast \varphi \in (Q - AP)(\mathbb{R} : X)\), \(\varphi \in D\) for a bounded vector-valued distribution \(T \in D'_{L^1}(X)\), in contrast with the considerations of I. Cioranescu [12] and C. Bouzar, F. Z. Tchouar [5], where the above inclusions are required to be valid only for the test functions belonging to the space \(D_0\). The main result of paper is Theorem 3.9, where we state an important structural characterization for the class of (Q-AP) vector-valued ultradistributions. The last section of paper is reserved for certain applications to systems of ordinary differential equations in distribution and ultradistribution spaces.

The standard notation is used throughout the paper. By \((X, \| \cdot \|)\) we denote a non-trivial complex Banach space. The abbreviations \(C_b(I : X)\) and \(C(K : X)\), where \(K\) is a non-empty compact subset of \(\mathbb{R}\), stand for the spaces consisting of all bounded continuous functions \(I \mapsto X\) and all continuous functions \(K \mapsto X\), respectively. Both spaces are Banach endowed with sup-norm. By \(C_0([0, \infty) : X)\) we denote the closed subspace of \(C_b([0, \infty) : X)\) consisting of functions vanishing at plus infinity.

We say that a continuous function \(f : \mathbb{R} \to X\) is (AAP) iff there is a function \(q \in C_0([0, \infty) : X)\) and an (AP) function \(g : \mathbb{R} \to X\) such that \(f(t) = g(t) + q(t), t \geq 0\). By \(AAP(\mathbb{R} : X)\), we denote the vector space consisting of all (AAP) functions. See [9], [13]- [14], [18], [23], [33] and references cited therein for more details on the subject.
Let $1 \leq p < \infty$. Then we say that a function $f \in L^p_{\text{loc}}(\mathbb{R} : X)$ is Stepanov $p$-bounded, $S^p$-bounded shortly, iff

$$
\|f\|_{S^p} := \sup_{t \in \mathbb{R}} \left( \int_t^{t+1} \|f(s)\|^p ds \right)^{1/p} < \infty.
$$

The space $L^p_S(\mathbb{R} : X)$ consisted of all $S^p$-bounded functions becomes a Banach space equipped with the above norm. A function $f \in L^p_S(\mathbb{R} : X)$ is said to be Stepanov $p$-almost periodic, $(S^p$-AP) shortly, iff there are two locally $p$-integrable functions $g : \mathbb{R} \to X$ and $q : [0, \infty) \to X$ satisfying the following conditions:

(i) $g(\cdot)$ is $(S^p$-AP),
(ii) $\hat{g}(\cdot)$ belongs to the class $C_0([0, \infty) : L^p([0, 1] : X))$,
(iii) $f(t) = g(t) + q(t)$ for all $t \geq 0$.

By $AAP^p_S(\mathbb{R} : X)$ we denote the space consisting of all $(S^p$-AAP) functions.

The notion of an almost automorphic function (AAut) on topological group was introduced and further analyzed in the landmark papers by W. A. Veech [31]- [32] between 1965 and 1967. For more details about (AP) and (AAut) functions with values in Banach spaces, we refer the reader to the monographs [13] by T. Diagana and [18] by G. M. N’Guérékata.

Following [25], we can recall the definition of (Q-AP) functions. Let we suppose that $I = [0, \infty)$ or $I = \mathbb{R}$. It is said that a bounded continuous function $f : I \to X$ is (Q-AP) iff for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$
\|f(t + \tau) - f(t)\| \leq \varepsilon, \quad \text{provided } t \in I \text{ and } |t| \geq M(\varepsilon, \tau). \quad (1.1)
$$

We denote by $Q - AP(I : X)$ the set consisting of all (Q-AP) functions from $I$ into $X$. It is not relevant whether we write (1.1) or $\|f(t + \tau) - f(t)\| \leq \varepsilon$, provided $t \in I$, $|t| \geq M(\varepsilon, \tau)$ and $|t + \tau| \geq M(\varepsilon, \tau)$. So, we can easily seen that the class $AAP(I : X)$ is contained in the class $Q - AAP(I : X)$, since the number $M$ depends only on $\varepsilon$ and not on $\tau$ for (AAP) functions. Further, we will use the shorthand:

(S): "there exists finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number”.

Let $f \in L^p_S(I : X)$. It is said that $f(\cdot)$ is $(S^p$-Q-AP), iff for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\varepsilon, \tau) > 0$ such
that
\[
\int_t^{t+1} \| f(s+t) - f(s) \|^p \, ds \leq \varepsilon^p,
\]
provided \( t \in I, |t| \geq M(\varepsilon, \tau). \) (1.2)

Denote by \( S^p Q - AP(I : X) \) the set consisting of all \((S^p-Q-AP)\) functions from \( I \) into \( X \). From the definition, follows that \( Q - AP(I : X) \subseteq S^p Q - AP(I : X) \). This inclusion is strict, since the function \( f(t) = \chi|_{[-1, \infty)}(t), t \in \mathbb{R} \) is in \( S^p Q - AP(\mathbb{R} : X) \) but not in class \( Q - AP(\mathbb{R} : X) \), because \( f(\cdot) \) is not continuous. Furthermore, any \((S^p-AAP)\) is \((S^p-Q-AP)\), so \( S^p AAP(I : X) \subseteq S^p Q - AP(I : X) \). If \( 1 \leq p < p' < \infty \), then \( S^p Q - AP(I : X) \subseteq S^p Q - AP(I : X) \) and for every function \( f \in L^p_S(I : X) \), we have that \( f(\cdot) \) is \((S^p-Q-AP)\) iff the function \( \tilde{f} : I \to L^p([-1, 1]) \) defined by \( \tilde{f}(t)(s) = f(t + s) \) is \((Q-AP)\).

Let us recall that a continuous function \( f : I \to X \) is said to be \((AAut)\) iff for every real sequence \((b_n)\) there exist a subsequence \((a_n)\) of \((b_n)\) and a map \( g : I \to X \) such that \( \lim_{n \to \infty} f(t + a_n) = g(t) \) and \( \lim_{n \to \infty} g(t + a_n) = f(t) \), pointwise for \( t \in I \). The space of all automorphic functions \( f : I \to X \) will be denoted by \( AA(I : X) \).

A bounded continuous function \( f : I \to X \) is said to be \((AAut)\) iff there exist two functions \( h \in AAut(I : X) \) and \( q \in C_0(I : X) \) such that \( f = h + q \) on \( I \). The notion of Stepanov \( p\)–almost automorphy \((S^p-AAut)\) has been introduced by G. M. N’Guérékata and A. Pankov in [19]: A function \( f \in L^p_{loc}(I : X) \) is called \((S^p-AAut)\) iff for every real sequence \((a_n)\), there exists a subsequence \((a_{n_k})\) and a function \( g \in L^p_{loc}(I : X) \) such that
\[
\lim_{k \to \infty} \int_t^{t+1} \left\| f(a_{n_k} + s) - g(s) \right\|^p \, ds = 0
\]
and
\[
\lim_{k \to \infty} \int_t^{t+1} \left\| g(s - a_{n_k}) - f(s) \right\|^p \, ds = 0
\]
for each \( t \in I \). The vector space of all \((S^p-AAut)\) functions will be denoted by \( S^p AAut(I : X) \). A function \( f \in L^p_{loc}(I : X) \) is called \((S^p-AAut)\) iff there exists an \((S^p-AAut)\) function \( g(\cdot) \) and a function \( q \in L^p_S(I : X) \) such that \( f(t) = g(t) + q(t), t \geq 0 \) and \( \tilde{q} \in C_0(I : L^p([0, 1] : X)) \). The vector space consisting of all \((S^p-AAut)\) functions will be denoted by \( S^p AAut(I : X) \).

Concerning distribution spaces, we will use the following elementary notion (cf. L. Schwartz [28] for more details). By \( D(X) = D(\mathbb{R} : X) \) we denote the Schwartz space of test functions with values in \( X \), by \( S(X) = S(\mathbb{R} : X) \) we denote the space of rapidly decreasing functions with values in \( X \), and by \( E(X) = E(\mathbb{R} : X) \) we denote the space of all infinitely differentiable functions with values in \( X \); \( D = D(\mathbb{C}) \), \( S = S(\mathbb{C}) \) and \( E = E(\mathbb{C}) \). Here \( S(\mathbb{C}) \) denotes the space of complex valued rapidly decreasing functions on \( \mathbb{R} \). The same for \( E(\mathbb{C}) \) and \( D(\mathbb{C}) \). The spaces of all linear continuous mappings from \( D \), \( S \) and \( E \) into \( X \) will be denoted by \( D'(X), S'(X) \) and \( E'(X) \), respectively. Set \( D_0 := \{ \phi \in D : \text{supp}(\phi) \subseteq [0, \infty) \} \).
1.1. Vector-valued ultradistributions

In this section the approach of Komatsu to the vector-valued ultradistributions will be followed, with the sequence $(M_p)$ of positive real numbers satisfying $M_0 = 1$ and the following conditions: (M.1): $M_p^2 \leq M_{p+1}M_{p-1}, \ p \in \mathbb{N}$, (M.2): $M_p \leq AH^p \sup_{0 < i \leq p} M_i M_{p-i}, \ p \in \mathbb{N}$, for some $A, H > 1$, (M.3): $\sum_{p=1}^{\infty} \frac{M^{-1}_{p-1}}{M_p} < \infty$. Any use of the condition (M.3): $\sup_{p \in \mathbb{N}} \sum_{q=p+1}^{\infty} \frac{M_{p+1}}{pM_q M_p} < \infty$, which is slightly stronger than (M.3'), will be explicitly emphasized.

Let us recall that the Gevrey sequence $(p^s), s > 1$, satisfies the above conditions. Set $m_p := \frac{M_p}{M_{p-1}}, p \in \mathbb{N}$.

The space of Beurling, resp., Roumieu ultradifferentiable functions, is defined by $D^{(M_p)} := \text{indlim}_{K \in \mathbb{R}} D^{(M_p)}_K$, resp., $D^{(M_p)}_K := \text{indlim}_{K \in \mathbb{R}} D^{(M_p)}_K$, where

$$D^{(M_p)}_K := \text{projlim}_{h \to 0} D^{(M_p)}_{K,h}, \ D^{(M_p)}_{K,h} := \{ \phi \in C^\infty(\mathbb{R}) : \text{supp}\phi \subseteq K, \|\phi\|_{M_p,h,K} < \infty \}$$

and

$$\|\phi\|_{M_p,h,K} := \sup \left\{ \frac{h^p|\phi(p)(t)|}{M_p} : t \in K, \ p \in \mathbb{N}_0 \right\}.$$ 

Henceforward, the asterisk * is used to denote both, the Beurling case $(M_p)$ or the Roumieu case $(M_p)$. Set $D^*_K := \{ \phi \in D^* : \text{supp}(\phi) \subseteq [0, \infty) \}$. The space consisted of all continuous linear functions from $D^*$ into $X$, denoted by $D^*(X) := L(D^* : X)$, is said to be the space of all $X$-valued ultradistributions of *-class. We also need the notion of space $E^*(X)$, defined as $E^*(X) := \text{indlim}_{K \in \mathbb{R}} E^*_K(X)$, where in Beurling case $E^{(M_p)}_K(X) := \text{projlim}_{h \to 0} E^{(M_p)}_{K,h}(X)$, resp., in Roumieu case $E^{(M_p)}_K(X) := \text{indlim}_{h \to 0} E^{(M_p)}_{K,h}(X)$, and

$$E^{(M_p)}_{K,h}(X) := \left\{ \phi \in C^\infty(\mathbb{R} : X) : \sup_{p \geq 0} \frac{h^p|\phi(p)||_{C(K:X)}}{M_p} < \infty \right\}.$$ 

The space consisting of all linear continuous mappings $E^*(\mathbb{C}) \to X$ is denoted by $E^{*}(X); E^{*} := E^{*}((\mathbb{C})$. Notation $E(\mathbb{C})$ means that we consider ultradifferentiable functions on $\mathbb{R}$ with values in $\mathbb{C}$. An entire function of the form $P(\lambda) = \sum_{p=0}^{\infty} a_p \lambda^p$, $\lambda \in \mathbb{C}, (a_p \text{ complex numbers as well})$ is of class $(M_p)$, resp., of class $(M_p)$, if there exist $l > 0$ and $C > 0$, resp., for every $l > 0$ there exists a constant $C > 0$, such that $|a_p| \leq Cl^p/M_p, \ p \in \mathbb{N}$ (20). The corresponding ultradifferential operator $P(D) = \sum_{p=0}^{\infty} a_p D^p$ is said to be of class $(M_p)$, resp., of class $(M_p)$; it acts as a continuous linear operator between the spaces $D^*$ and $D^*$ ($D^*$ and $D^*$). The convolution of Banach space valued ultradistributions and scalar-valued ultradifferentiable functions of the same class will be taken in the sense of considerations given on page 685 of [22]. Let remind ourselves that, for every $f \in D^*(X)$ and
φ ∈ D∗, we have \( f * φ ∈ E^*(X) \) as well as that the linear mapping \( φ ↦ ∗φ : D^*(X) → E^*(X) \) is continuous. The convolution of an \( X \)-valued ultradistribution \( f(·) \) and a scalar-valued ultradistribution \( g ∈ E^* \) with compact support, defined by the identity \( \langle f * g, φ \rangle = \langle f, g(·) * φ \rangle \), where \( g(x) = g(-x) \) and \( φ \) is a test function in \( D^* \), is an \( X \)-valued ultradistribution and the mapping \( g ∗ · : D^*(X) → D^*(X) \) is continuous. Set \( (T_h, φ) := \langle T, φ(· − h) \rangle \), \( T ∈ D^*(X) \), \( φ ∈ D^* (h > 0) \). We will use a similar definition for vector-valued distributions.

Assume that the sequence \( (M_p) \) satisfies (M.1), (M.2) and (M.3). Then

\[
P_t(x) = (1 + x^2) \prod_{p ∈ \mathbb{N}} \left( 1 + \frac{x^2}{t^2 m_p^2} \right),
\]

resp.

\[
P_{r_p}(x) = (1 + x^2) \prod_{p ∈ \mathbb{N}} \left( 1 + \frac{x^2}{r_p^2 m_p^2} \right),
\]

defines an ultradifferential operator of class \( (M_p) \), resp., of class \( \{M_p\} \); here, \( (r_p) \) is a sequence of positive real numbers tending to infinity. The family consisting of all such sequences will be denoted by \( R \) henceforth. For more details on the subject, the reader may consult [20]-[22].

The spaces of tempered ultradistributions of Beurling, resp., Roumieu type, are defined by S. Pilipović [27] as duals of the corresponding test spaces

\[
S^{(M_p)} := \text{projlim}_{h → 0} S^{M_p, h}, \quad \text{resp.,} \quad S^{\{M_p\}} := \text{indlim}_{h → 0} S^{M_p, h},
\]

where

\[
S^{M_p, h} := \{ φ ∈ C^∞(\mathbb{R}) : \|φ\|_{M_p, h} < \infty \} \quad (h > 0),
\]

\[
\|φ\|_{M_p, h} := \sup \left\{ \frac{h^{α+β}}{M^α M^β} (1 + t^2)^{β/2} |φ^(α)(t)| : t ∈ \mathbb{R}, \ α, β ∈ \mathbb{N}_0 \right\}.
\]

2. Quasi-Asymptotical Almost Periodicity of Vector-Valued Distributions

We refer the reader to [5], [10] and [26] for the basic results about vector-valued (AP) distributions. Let \( 1 ≤ p ≤ ∞ \). Then by \( D_L(X) \) we denote the vector space consisting of all infinitely differentiable functions \( f : \mathbb{R} → X \) satisfying that for each number \( j ∈ \mathbb{N}_0 \) we have \( f^{(j)} ∈ L^p(\mathbb{R} : X) \). The Fréchet topology on \( D_L(X) \) is induced by the following system of seminorms

\[
\|f\|_k := \sum_{j=0}^{k} \|f^{(j)}\|_{L^p(\mathbb{R})}, \quad f ∈ D_L(X) \quad (k ∈ \mathbb{N}).
\]

If \( X = \mathbb{C} \), then the above space is simply denoted by \( D_L \). A linear continuous mapping \( f : D_L → X \) is said to be a bounded \( X \)-valued distribution; the space consisting of such vector-valued distributions will be denoted by \( D'_L(X) \). Endowed with the strong topology, \( D'_L(X) \) becomes a complete locally convex space. For every \( f ∈ D'_L(X) \), we have that \( f|_S : S → X \) is a tempered \( X \)-valued distribution ([24]).
Theorem 2.1. The space of bounded vector-valued distributions tending to zero at plus infinity, \( B'_{+,0}(X) \) for short, is defined by
\[
B'_{+,0}(X) := \left\{ T \in \mathcal{D}'_L(X) : \lim_{h \to +\infty} \langle T_h, \varphi \rangle = 0 \text{ for all } \varphi \in \mathcal{D} \right\}.
\]
It can be simply verified that the structural characterization for the space \( B'_{+,0}(\mathbb{C}) \), (scalar valued case usually we will denote by \( B'_{+,0} \)) proved in [12, Proposition 1], is still valid in vector-valued case as well that the space \( B'_{+,0}(\mathbb{C}) \) is closed under differentiation.

Let \( T \in \mathcal{D}'_L(X) \). Then the following assertions are equivalent ([26]):

(i) \( T \ast \varphi \in AP(\mathbb{R} : X), \varphi \in \mathcal{D}, \text{ resp., } T \ast \varphi \in AA(\mathbb{R} : X), \varphi \in \mathcal{D} \).
(ii) There exist an integer \( k \in \mathbb{N} \) and (AP), resp. (AAut) functions
\[
f_j(\cdot) : [0, \infty) \to X (1 \leq j \leq k) \text{ such that } T = \sum_{j=0}^k f_j(j).
\]

We say that a distribution \( T \in \mathcal{D}'_L(X) \) is (AP), resp. (AAut), if \( T \) satisfies any of the above two equivalent conditions. By \( B'_{AP}(X) \), \( B'_{AA}(X) \) we denote the space consisting of all (AP), resp. (AAut), distributions.

**Definition 2.1.** A distribution \( T \in \mathcal{D}'_L(X) \) is said to be (AAP), resp. (AAAut), if there exist an (AP), resp. (AAut), distribution \( T_{ap} \in B'_{AP}(X), \text{ resp. } T_{aa} \in B'_{AA}(X) \), and a bounded distribution tending to zero at plus infinity \( Q \in B'_{+,0}(X) \) such that \( \langle T, \varphi \rangle = \langle T_{ap}, \varphi \rangle + \langle Q, \varphi \rangle, \varphi \in \mathcal{D}_0, \text{ resp. } \langle T, \varphi \rangle = \langle T_{aa}, \varphi \rangle + \langle Q, \varphi \rangle, \varphi \in \mathcal{D}_0 \).

By \( B'_{AAP}(X) \), resp. \( B'_{AAAut}(X) \), we denote the vector space consisting of all (AAP), resp. (AAAut) distributions.

It is well known that the representation \( T = T_{ap} + Q \) is unique in almost periodic case.

**Definition 2.2.** A distribution \( T \in \mathcal{D}'_L(X) \) is said to be (Q-AP) distribution if \( T \ast \varphi \in Q-AP(\mathbb{R} : X) \), for all \( \varphi \in \mathcal{D} \). The space of all (Q-AP) distributions will be denoted by \( \mathcal{D}'_{Q-AP}(X) \).

**Theorem 2.1.** The following statements hold:

i) \( \mathcal{D}'_{Q-AP}(X) \cap \mathcal{D}'_{AAAut}(X) = \mathcal{D}'_{AAP}(X) \);
\[
[\mathcal{D}'_{AAAut}(X) \setminus \mathcal{D}'_{AAP}(X)] \cap \mathcal{D}'_{Q-AP}(X) = \emptyset;
\]
ii) \( \mathcal{D}'_{AA}(X) \cap \mathcal{D}'_{Q-AP}(X) = \mathcal{D}'_{AP}(X) \).

**Proof.** The proof use the same arguments like in [25]. For completeness, we will give in sequel.

Let us consider only the case when \( I = \mathbb{R} \). The other case \( I = [0, \infty) \) is analogous. Since \( \mathcal{D}'_{AAP}(X) \subseteq \mathcal{D}'_{AAAut}(X) \cap \mathcal{D}'_{Q-AP}(X) \), we will prove the opposite inclusion.

Let \( T \in \mathcal{D}'_{AAAut} \cap \mathcal{D}'_{Q-AP}(X) \) and \( \varphi \in \mathcal{D} \) is arbitrary. Then \( T \ast \varphi \in AAAut(\mathbb{R} : X) \cap Q-AP(\mathbb{R} : X) \). Since \( T \ast \varphi \in AAAut(\mathbb{R} : X) \), there exist two functions \( f \in AAut(\mathbb{R} : X) \) and \( g \in C_0(\mathbb{R} : X) \), such that \( T \ast \varphi = f + g \) on \( \mathbb{R} \) and for every \( \varepsilon > 0 \),
Let $C(S)$ holds and there exists a number $M(\varepsilon, \tau) > 0$ such that
\[
\| (f(t + \tau) - f(t)) + (g(t + \tau) - g(t)) \| \leq \varepsilon, \quad \text{for } t \in \mathbb{R} \text{ and } |t| \geq M(\varepsilon, \tau). \tag{2.1}
\]

Let $\varepsilon > 0$ be fixed and the real number $\tau$ satisfies (2.1) for $|\tau| \geq M(\varepsilon, \tau)$. By $g \in C_0(\mathbb{R} : X)$, there exists a finite number $M_1(\varepsilon, \tau) \geq M(\varepsilon, \tau)$ such that $\|g(t + \tau) - g(t)\| \leq \frac{\varepsilon}{2}$ for $t \in \mathbb{R}$, $|t| \geq M_1(\varepsilon, \tau)$. We define the function $F : \mathbb{R} \rightarrow X, F(t) = f(t + \tau) - f(t), t \in \mathbb{R}$. The space $AAut(\mathbb{R} : X)$ is translation invariant, $F \in AAut(\mathbb{R} : X)$. Using supremum formula, we have
\[
\sup_{t \in \mathbb{R}} \|F(t)\| = \sup_{t \geq M_1(\varepsilon, \tau)} \|F(t)\| = \sup_{t \geq M_1(\varepsilon, \tau)} \|f(t + \tau) - f(t)\| \leq \frac{\varepsilon}{2}.
\]
So, $\|f(t + \tau) - f(t)\| \leq \frac{\varepsilon}{2}$ for all $t \in \mathbb{R}$, so $f$ is (AP) function. Hence, $D_{AAP}(X) \subseteq D_{Q-AP}(X)$, so the equation in $(i)$ holds. The equation in $(ii)$, follows from the proof of $(i)$. \hfill \Box

**Definition 2.3.** Let $\omega \in I$.

a) A bounded continuous function $f : I \rightarrow X$ is said to be asymptotically $\omega$-almost periodic ($AAP_\omega$ in short) if there exist a function $g \in C_0(I : X)$ and a function $q \in C_0(I : X)$ such that $f(t) = g(t) + q(t)$ for all $t \in I$;

b) A bounded continuous function $f : I \rightarrow X$ is said to be $S$-asymptotically $\omega$-periodic ($S-AP_\omega$ in short) if $\lim_{|t| \rightarrow \infty} \|f(t + \omega) - f(t)\| = 0$. Denote by $S-AP_\omega(I : X)$ the space consisting of all such functions;

c) A Stepanov $p$-bounded function $f(\cdot)$ is said to be Stepanov $p$-asymptotically $\omega$-periodic ($S^p-AP_\omega$ in short) if
\[
\lim_{|t| \rightarrow \infty} \int_t^{t+1} \|f(s + \omega) - f(s)\|^p ds = 0.
\]

The space of all $S^p-AP_\omega$ functions is denoted by $S^p-AP_\omega(I : X)$. Note that $S-AP_\omega(I : X) \subseteq S^p-AP_\omega(I : X)$.

**Definition 2.4.** Let $\omega \in I$ and $T \in D_{\omega}^I(X)$.

a) A distribution $T$ is said to be $AAP_\omega$ if $T * \varphi \in AAP_\omega(I : X)$ for every $\varphi \in \mathcal{D}$. The space of all asymptotically $\omega$-almost periodic distributions is denoted by $D_{AAP_\omega}(X)$.

b) A distribution $T$ is said to be $S^p-AP_\omega$ if $T * \varphi \in S^p-AP_\omega(I : X)$ for every $\varphi \in \mathcal{D}$. The space of all $S^p-AP_\omega$ distributions is denoted by $D_{S^p-AP_\omega}(X)$.

c) A distribution $T$ is said to be $S-AP_\omega$ if $T * \varphi \in S-AP_\omega(I : X)$, for every $\varphi \in \mathcal{D}$. The space of all $S-AP_\omega$ distributions is denoted by $D_{S-AP_\omega}(X)$.

d) A distribution $T$ is said to be $S^p-AAut$ if $T * \varphi \in S^p-AAut(I : X)$, for every $\varphi \in \mathcal{D}$. The space of all $S^p-AAut$ distributions is denoted by $D_{S^p-AAut}(X)$.

e) A distribution $T$ is said to be $S^p-AAut$ if $T * \varphi \in S^p-AAut(I : X)$, for every $\varphi \in \mathcal{D}$.
Definition 2.5. Let \( T \in \mathcal{D}_I'(X) \).

a) A distribution \( T \) is said to be \((SpQ - AP)\) if \( T * \varphi \in S^0 Q - AP(I : X) \), for every \( \varphi \in \mathcal{D} \). The space of all \((SpQ - AP)\) distributions is denoted by \( \mathcal{D}'_{S^0 Q - AP}(X) \).

b) A distribution \( T \) is said to be \((SQ - AP)\) if \( T * \varphi \in SQ - AP_\omega(I : X) \), for every \( \varphi \in \mathcal{D} \). The space of all \((SQ - AP)\) distributions is denoted by \( \mathcal{D}'_{SQ - AP_\omega}(X) \).

c) A distribution \( T \) is said to be \((Q - AP)\) if \( T * \varphi \in Q - AP_\omega(I : X) \), for every \( \varphi \in \mathcal{D} \). The space of all \((Q - AP)\) distributions is denoted by \( \mathcal{D}'_{Q - AP_\omega}(X) \).

d) A distribution \( T \) is said to be \((SpQ - AP_\omega)\) if \( T * \varphi \in S^0 Q - AP_\omega(I : X) \), for every \( \varphi \in \mathcal{D} \). The space of all \((SpQ - AP_\omega)\) distributions is denoted by \( \mathcal{D}'_{S^0 Q - AP_\omega}(X) \).

e) A distribution \( T \) is said to be \((SpQ - P')\) if \( T * \varphi \in S^0 Q - P'(I : X) \), for every \( \varphi \in \mathcal{D} \). The space of all \((SpQ - P')\) distributions is denoted by \( \mathcal{D}'_{S^0 Q - P'}(X) \).

The next theorem follows from [25, Proposition 2.7].

Theorem 2.2. Let \( \omega \in I \). It holds \( \mathcal{D}'_{S - AP_\omega}(X) \subseteq \mathcal{D}'_{Q - AP}(X) \).

Proof. Let \( \varepsilon > 0 \) be given and \( T \in \mathcal{D}'_{S - AP_\omega}(X) \). Take \( L(\varepsilon) = 2n \). Then for any interval \( I' \subseteq I \) of length \( L(\varepsilon) \) contains a number \( \tau = n\omega > 0 \) such that

\[
\|T * \varphi(t + \omega) - T * \varphi(t)\| < \frac{\varepsilon}{2\omega}, \quad \text{for} \quad |t| \leq M(\varepsilon, n).
\]

Hence,

\[
\|T * \varphi(t + n\omega) - T * \varphi(t)\| \leq \sum_{k=0}^{n-1} \|f(t + \tau - k\omega) - f(t + \omega - (k + 1)\omega)\| \leq \frac{n\varepsilon}{n\omega} = \varepsilon,
\]

for \( |t| \geq M(\varepsilon, n) + n\omega \), so the conclusion of the theorem follows. \( \Box \)

Theorem 2.3. Let \( \omega \in I \). Then

\[ \mathcal{D}'_{S - AP_\omega}(X) \cap \mathcal{D}'_{AAAut}(X) \subseteq \mathcal{D}'_{AP_\omega}(X). \]

Proof. Let \( T \in \mathcal{D}'_{S - AP_\omega}(X) \cap \mathcal{D}'_{AAAut}(X) \) and \( \varphi \in \mathcal{D} \). Then by the definitions of \( \mathcal{D}'_{S - AP_\omega}(X) \) and \( \mathcal{D}'_{AAAut}(X) \), \( T * \varphi \in S - AP_\omega(I : X) \cap AAAut(I : X) \). Since, \( T * \varphi \in AAAut(I : X) \), there exist \( g \in AAAut(I : X) \) and \( h \in C_0([0, \infty) : X) \) such that \( T * \varphi = g + h \). It is sufficient to prove that \( g \in P_\omega(I : X) \). Since, \( C_0([0, \infty) : X) \subseteq S - AP_\omega(X) \) it follows that \( g = T * \varphi - h \in S - AP_\omega(X) \). Hence,

\[ \lim_{t \to \infty} \|g(t + \omega) - g(t)\| = 0. \quad (2.2) \]

Now, since \( g \) is \((AAut)\), we can find a subsequence \((t_k)\) of \((t_n)\) such that for all \( t \in \mathbb{R} \),

\[ \lim_{m \to \infty} \lim_{k \to \infty} \|g(t + \omega - t_{n_k} - t_{n_m}) - g(t + t_{n_k} - t_{n_m}) - g(t + \omega) - g(t)\| = 0. \]

From (2.2) and \( \lim_{k \to \infty} (t_{n_k} - t_{n_m}) = +\infty \), we have

\[ \lim_{k \to \infty} g(t + \omega + t_{n_k} - t_{n_m}) - g(t + t_{n_k} - t_{n_m}) = 0, \]

for all \( t \in I \) so \( g(t + \omega) - g(t) = 0 \), for all \( t \in I \). This implies \( g(t + \omega) - g(t) = 0 \) for all \( t \in I \), so the proof is finished. \( \Box \)
Now, using the technique in the previous result we can give the following:

**Theorem 2.4.** The following statements hold:

i) \( \mathcal{D}'_{\mathcal{SP}_{-AAP}}(X) \cap \mathcal{D}'_{\mathcal{SP}_{Q-AP}}(X) = \mathcal{D}'_{\mathcal{SP}_{AAP}}(X) \)

\[\mathcal{D}'_{\mathcal{SP}_{-AAP}}(X) \setminus \mathcal{D}'_{\mathcal{SP}_{AAP}}(X) = \emptyset;\]

ii) \( \mathcal{D}'_{\mathcal{SP}_{-AAP}}(X) \cap \mathcal{D}'_{\mathcal{SP}_{Q-AP}}(X) = \mathcal{D}'_{\mathcal{SP}_{AAP}}(X).\)

The proof is quite similar like of Theorem 2.3, so we omit it.

**Theorem 2.5.** It holds that \( \mathcal{D}'_{\mathcal{SP}_{-AP}}(X) \subseteq \mathcal{D}'_{\mathcal{SP}_{Q-AP}}(X). \)

**Theorem 2.6.** Let \( T \in \mathcal{D}'_{L^1}(X). \) The then following holds:

i) Let \( T \in \mathcal{D}'_{Q-AP}(X), \) resp., \( T \in \mathcal{D}'_{SP_{Q-AP}}(X). \) Then \( cT \) is a \((Q-AP)\) distribution, resp., \((SP)^\mathcal{Q}-AP\) distribution, for any \( c \in \mathbb{C};\)

ii) If \( (T_n) \) is a sequence in \( \mathcal{D}'_{Q-AP}(X), \) resp., \( \mathcal{D}'_{SP_{Q-AP}}(X) \) and \( T_n \to T \) uniformly in \( \mathcal{D}'_{Q-AP}(X), \) resp., \( \mathcal{D}'_{SP_{Q-AP}}(X) \), then \( T \in \mathcal{D}'_{Q-AP}(X), \) resp., \( T \in \mathcal{D}'_{SP_{Q-AP}}(X); \)

iii) Any translation \( T_h = \langle T, \varphi \rangle \) of \( T \in \mathcal{D}'_{Q-AP}(X) \) \( (T \in \mathcal{D}'_{SP_{Q-AP}}(X)) \) is again in \( \mathcal{D}'_{Q-AP}(X) \) \( (\mathcal{D}'_{SP_{Q-AP}}(X)) \).

**Proof.** The case when \( T \in \mathcal{D}'_{SP_{Q-AP}}(X) \) is similar with the case when \( T \in \mathcal{D}'_{Q-AP}(X), \) so we skip it.

i) Let \( T \in \mathcal{D}'_{Q-AP}(X) \) and \( \varphi \in \mathcal{D}. \) Then \( T \ast \varphi \in Q-AP(I : X). \) Since, \( (cT) \ast \varphi = c(T \ast \varphi) \), using [25, Theorem 2.13 (i)], the statement of i) follows.

ii) Let \( \varepsilon > 0 \) be given. Then there exists a distribution \( T_{n_0} \) such that \( |T \ast \varphi(x) - T_{n_0} \ast \varphi(x)| < \frac{\varepsilon}{3}, \) for all \( x \in I, |x| \geq M(\varepsilon, \tau), \) for some \( M(\varepsilon, \tau) > 0 \) and for all \( \varphi \in \mathcal{D}. \)

Let \( E_q(\varepsilon,T) \), be the set of all translation numbers of \( T \) belonging to \( \varepsilon \geq 0 \) if \( |T \ast \varphi(x + \tau) - T \ast \varphi(x)| \leq \varepsilon, \) for all \( \varphi \in \mathcal{D}. \) Put \( \tau \in E\left(\frac{1}{5}\varepsilon, T_{n_0} \ast \varphi(x)\right). \) Then,

\[ |T \ast \varphi(x + \tau) - T \ast \varphi(x) | \leq |T \ast \varphi(x + \tau) - T_{n_0} \ast \varphi(x + \tau) |
\]
\[ + |T_{n_0} \ast \varphi(x + \tau) - T_{n_0} \ast \varphi(x) | + |T_{n_0} \ast \varphi(x) - T \ast \varphi(x) | \leq \varepsilon. \]

Since \( E_q(\varepsilon,T \ast \varphi(x)) \supset E_q\left(\frac{1}{5}\varepsilon, T_{n_0} \ast \varphi(x)\right) \), it follows \( E_q(\varepsilon,T \ast \varphi(x)) \) is relatively dense and \( \varepsilon > 0 \) is arbitrary, we conclude that \( T \) is \((Q-AP)\) distribution.

iii) Let \( T \in \mathcal{D}'_{Q-AP}(X) \) and \( \varphi \in \mathcal{D} \) Then \( T \ast \varphi \in Q-AP(I : X). \) Now, since \( T_h \ast \varphi = \langle T, \varphi(\cdot - h) \rangle = (\tau_h T) \ast \varphi = \tau_h(T \ast \varphi), \) by [25, Theorem 2.13 (v)], \( \tau_h(T \ast \varphi) \in Q-AAP(I : X). \) Hence \( T_h \in \mathcal{D}'_{Q-AP}(X). \)

Further on, we would like to observe that the following structural result holds in vector-valued case:

**Theorem 2.7.** Let \( T \in \mathcal{D}'_{L^1}(X). \) Then the following assertions are equivalent:

i) \( T \in \mathcal{D}'_{Q-AP}(X). \)

ii) There exist an integer \( k \in \mathbb{N} \) and \((Q-AP)\) functions \( f_j(\cdot) : \mathbb{R} \to X \) \( (0 \leq j \leq k) \)

such that \( T = \sum_{j=0}^k f_j(\cdot) \) on \([0, \infty). \)
(iii) There exist \( S \in \mathcal{D}'_{L^1}(X), k \in \mathbb{N} \) and bounded (Q-AP) functions \( f_j(\cdot) : \mathbb{R} \to X \) \((0 \leq j \leq k)\) such that \( S = \sum_{j=0}^{k} f_j^{(j)} \) on \( \mathbb{R} \), and \( \langle S, \varphi \rangle = \langle T, \varphi \rangle \) for all \( \varphi \in \mathcal{D}_0 \).

(iv) There exists \( S \in \mathcal{D}'_{L^1}(X) \) such that \( \langle S, \varphi \rangle = \langle T, \varphi \rangle \) for all \( \varphi \in \mathcal{D}_0 \) and \( S * \varphi \in Q - AP(\mathbb{R} : X), \varphi \in \mathcal{D} \).

**Proof.** The equivalence of (i)-(ii) can be proved as in (Q-AP) scalar-valued case (see [12, Theorem I, Proposition 1]).

The implication (ii) \( \Rightarrow \) (iii) trivially follows from the fact that the expression \( S = \sum_{j=0}^{k} f_j^{(j)} \) defines an element from \( \mathcal{D}'_{L^1}(X) \). Since the space \( A \equiv Q - AP(\mathbb{R} : X) \cap C_b(\mathbb{R} : X) \) is uniformly closed (and therefore, \( C^\infty\)-uniformly closed), closed under addition and \( A * \mathcal{D} \subseteq A \) (see [1] for the notion), so we have that (iii) implies (iv).

The implication (iv) \( \Rightarrow \) (ii) is trivial, hence we have the equivalence of assertions (i)-(iv).

Concerning the assertions of Theorem 2.7, it is worth noting the following:

**Remark 2.1.** The validity of (iv) for some \( S \in \mathcal{D}'_{L^1}(X) \) implies its validity for \( S \) replaced therein with \( S_Q = S + Q \), where \( Q \in \mathcal{D}'_{L^1}(X) \) and \( \text{supp}(Q) \subseteq (-\infty, 0] \).

For this, it suffices to observe that \( (Q * \varphi)(x) = \langle Q, \varphi(x - \cdot) \rangle = 0 \) for all \( x \geq \text{sup}(\text{supp}(\varphi)), \varphi \in \mathcal{D} \).

3. **QUASI-ASYMPTOTICAL ALMOST PERIODICITY OF VECTOR-VALUED ULTRADISTRIBUTIONS**

For any \( h > 0 \), we define

\[
\mathcal{D}^{M_p, h}_{L^1} := \left\{ f \in \mathcal{D}_{L^1} : \| f \|_{1,h} := \sup_{p \in \mathbb{N}_0} h^p \| f^{(p)} \|_1 < \infty \right\}.
\]

Then \( (\mathcal{D}^{M_p, h}_{L^1}, \| \cdot \|_{1,h}) \) is a Banach space and the space of all \( X \)-valued bounded Beurling ultradistributions of class \( \{M_p\} \), resp., \( X \)-valued bounded Roumieu ultradistributions of class \( \{M_p\} \), is defined to be the space consisting of all linear continuous mappings from \( \mathcal{D}^{(M_p)}_{L^1} \), resp., \( \mathcal{D}^{(M_p)}_{L^1} \), into \( X \), where

\[
\mathcal{D}^{(M_p)}_{L^1} := \text{projlim}_{h \to +\infty} \mathcal{D}^{M_p, h}_{L^1},
\]

resp.,

\[
\mathcal{D}^{(M_p)}_{L^1} := \text{indlim}_{h \to 0+} \mathcal{D}^{M_p, h}_{L^1}.
\]

These spaces, equipped with the strong topologies, will be shortly denoted by \( \mathcal{D}^{(M_p)}_{L^1}(X) \), resp., \( \mathcal{D}^{(M_p)}_{L^1}(X) \). We will use the shorthand \( \mathcal{D}^{(M_p)}_{L^1}(X) \) to denote either \( \mathcal{D}^{(M_p)}_{L^1}(X) \) or \( \mathcal{D}^{(M_p)}_{L^1}(X) \); \( \mathcal{D}^{(M_p)}_{L^1}(X) \equiv \mathcal{D}^{(M_p)}_{L^1}(\mathbb{C}) \). As it can be easily proved, \( \mathcal{D}^{(M_p)}_{L^1}(X) \) is a complete locally convex space.

It is well known that \( \mathcal{D}^{(M_p)}_{L^1} \), resp., \( \mathcal{D}^{(M_p)}_{L^1} \), is a dense subspace of \( \mathcal{D}^{(M_p)}_{L^1} \), resp., \( \mathcal{D}^{(M_p)}_{L^1} \), as well as that \( \mathcal{D}^{(M_p)}_{L^1} \subseteq \mathcal{D}^{(M_p)}_{L^1} \). It can be simply proved that \( f|_{\mathcal{S}(M_p)} : \)
\[ S(M_p) \to X, \ \text{resp.}, \ f|_{S(M_p)} : S(M_p) \to X, \] is a tempered \( X \)-valued ultradistribution of class \( (M_p) \), resp., of class \( \{M_p\} \). The space \( D^{*}_{\lambda_1}(X) \) is closed under the action of ultradifferential operators of \( * \)-class.

The space of bounded vector-valued ultradistributions tending to zero at plus infinity, \( D^{*}_{+,0}(X) \) for short, is defined by

\[
D^{*}_{+,0}(X) := \left\{ T \in D^{*}_{\lambda_1}(X) : \lim_{h \to +\infty} \langle T_h, \phi \rangle = 0 \text{ for all } \phi \in D^* \right\}.
\]

Let \( T \in D^{*}_{\lambda_1}(X) \). Then we say that \( T \) is (AP), resp. (AAut), if \( T \) satisfies: \( T \ast \phi \in AP(\mathbb{R} : X) \), \( \phi \in D^* \), resp., \( T \ast \phi \in AAut(\mathbb{R} : X) \), \( \phi \in D^* \). By \( D^{*}_{AP}(X) \), we denote the vector space consisting of all almost periodic ultradistributions of \( * \)-class.

**Definition 3.1.** An ultradistribution \( T \in D^{*}_{\lambda_1}(X) \) is said to be (AAP) if there exist an (AP) ultradistribution \( T_{ap} \in D^{*}_{AP}(X) \), and a bounded ultradistribution tending to zero at plus infinity \( Q \in D^{*}_{+,0}(X) \) such that \( \langle T, \phi \rangle = \langle T_{ap}, \phi \rangle + \langle Q, \phi \rangle \), \( \phi \in D^* \).

By \( D^{*}_{AAP}(X) \), we denote the vector space consisting of all (AAP) ultradistributions of \( * \)-class.

Likewise in distribution case, decomposition of an (AAP) ultradistribution of \( * \)-class into its (AP) part and bounded, tending to zero at plus infinity part, is unique. The space \( D^{*}_{AAP}(X) \) is closed under the action of ultradifferential operators of \( * \)-class. This follows from the fact that this is true for the space \( D^{*}_{AP}(X) \), (see [24] and [26]), as well as that, for every \( Q \in D^{*}_{+,0}(X) \) and for every ultradifferential operator \( P(D) \) of \( * \)-class, we have \( \langle P(D)Q, \phi(\cdot - h) \rangle = \langle Q, [P(D)\phi](\cdot - h) \rangle \), \( h \in \mathbb{R} \).

For the sequel, we need the following preparation. Let \( A \subseteq D^*(X) \). Following B. Basit and H. Güenzler [1], whose examinations have been carried out in distributional case, we have recently introduced the following notion in [24]:

\[
D^{*}_{A}(X) := \left\{ T \in D^*(X) : T \ast \phi \in A \text{ for all } \phi \in D^* \right\}.
\]

Then \( D^{*}_{A}(X) = D^{*}_{A \cap \mathbb{R}}(X) \), for any set \( B \subseteq L^{1}_{loc}(\mathbb{R} : X) \) that contains \( C^{\infty}(\mathbb{R} : X) \), as well as the set \( D^{*}_{A}(X) \) is closed under the action of ultradifferential operators of \( * \)-class. Furthermore, the following holds [24]:

(i) Assume that there exist an ultradifferential operator \( P(D) = \sum_{p=0}^{\infty} a_p D^p \) of class \( (M_p) \), resp., of class \( \{M_p\} \), and \( f, g \in D^*_A(X) \) such that \( T = P(D)f + g \). If \( A \) is closed under addition, then \( T \in D^*_A(X) \).

(ii) If \( A \cap C(\mathbb{R} : X) \) is closed under uniform convergence, \( T \in D^{*}_{\lambda_1}((M_p) : X) \) and \( T \ast \phi \in A \), \( \phi \in D(M_p) \), then there is a number \( h > 0 \) such that for each compact set \( K \subseteq \mathbb{R} \) we have \( T \ast \phi \in A \), \( \phi \in D^*_K(M_p,h) \).

(iii) Assume that \( T \in D^{*(M_p)}(X) \) and there exists \( h > 0 \) such that for each compact set \( K \subseteq \mathbb{R} \) we have \( T \ast \phi \in A \), \( \phi \in D^*_K(M_p,h) \). If \( (M_p) \) additionally satisfies (M.3), then there exist \( l > 0 \) and two elements \( f, g \in A \) such that \( T = P(D)f + g \).
**Definition 3.2.** An ultradistribution \( T \in \mathcal{D}'_{\omega}(X) \) is said to be (Q-AP) ultradistribution if \( T \ast \phi \in Q - AP(\mathbb{R} : X) \), of class * for all \( \phi \in \mathcal{D}^* \). The space of all (Q-AP) ultradistributions will be denoted by \( \mathcal{D}'_{Q - AP}(X) \).

**Definition 3.3.** Let \( \omega \in I \) and \( T \in \mathcal{D}'_{\omega}(X) \).

1. An ultradistribution \( T \) is said to be \( (AP_\omega) \) if \( T \ast \phi \in AP_\omega(I : X) \) for every \( \phi \in \mathcal{D}^* \). The space of all \( (AP_\omega) \) ultradistributions is denoted by \( \mathcal{D}'_{AP_\omega}(X) \).
2. An ultradistribution \( T \) is said to be \( (SQ) \) if \( T \ast \phi \in SQ - AP_\omega(I : X) \) for every \( \phi \in \mathcal{D}^* \). The space of all \( (SQ) \) ultradistributions is denoted by \( \mathcal{D}'_{SQ - AP_\omega}(X) \).
3. An ultradistribution \( T \) is said to be \( (Sp) \) if \( T \ast \phi \in Sp - AP_\omega(I : X) \) for every \( \phi \in \mathcal{D}^* \). The space of all \( (Sp) \) ultradistributions is denoted by \( \mathcal{D}'_{Sp - AP_\omega}(X) \).
4. An ultradistribution \( T \) is said to be \( (SpQ) \) if \( T \ast \phi \in AQAut - AP_\omega(I : X) \), for every \( \phi \in \mathcal{D}^* \). The space of all \( (SpQ) \) ultradistributions is denoted by \( \mathcal{D}'_{SpQ - AP_\omega}(X) \).
5. An ultradistribution \( T \) is said to be \( (AAut) \) if \( T \ast \phi \in AAut - AP_\omega(I : X) \), for every \( \phi \in \mathcal{D}^* \). The space of all \( (AAut) \) ultradistributions is denoted by \( \mathcal{D}'_{AAut - AP_\omega}(X) \).

**Definition 3.4.** Let \( T \in \mathcal{D}'_{\omega}(X) \).

1. An ultradistribution \( T \) is said to be \( (SpQ - AP) \) if \( T \ast \phi \in SpQ - AP(I : X) \), for every \( \phi \in \mathcal{D}^* \). The space of all \( (SpQ - AP) \) ultradistributions is denoted by \( \mathcal{D}'_{SpQ - AP}(X) \).
2. An ultradistribution \( T \) is said to be \( (SQ - AP) \) if \( T \ast \phi \in SQ - AP_\omega(I : X) \), for every \( \phi \in \mathcal{D}^* \). The space of all \( (SQ - AP) \) ultradistributions is denoted by \( \mathcal{D}'_{SQ - AP_\omega}(X) \).
3. An ultradistribution \( T \) is said to be \( (Q - AP) \) if \( T \ast \phi \in Q - AP_\omega(I : X) \), for every \( \phi \in \mathcal{D}^* \). The space of all \( (Q - AP) \) ultradistributions is denoted by \( \mathcal{D}'_{Q - AP_\omega}(X) \).
4. An ultradistribution \( T \) is said to be \( (SpQ - AQAut) \) if \( T \ast \phi \in SpQ - AQAut(I : X) \), for every \( \phi \in \mathcal{D}^* \). The space of all \( (SpQ - AQAut) \) ultradistributions is denoted by \( \mathcal{D}'_{SpQ - AQAut}(X) \).
5. An ultradistribution \( T \) is said to be \( (SpQ - AAut) \) if \( T \ast \phi \in SpQ - AAut(I : X) \), for every \( \phi \in \mathcal{D}^* \). The space of all \( (SpQ - AAut) \) ultradistributions is denoted by \( \mathcal{D}'_{SpQ - AAut}(X) \).

Like in the case of distributions we can give the following theorems with the analogous proofs.

**Theorem 3.1.** The following statements hold:

1. \( \mathcal{D}'_{Q - AP}(X) \cap \mathcal{D}'_{AAut - AP_\omega}(X) = \mathcal{D}'_{AAut - AP_\omega}(X) \);
2. \( [\mathcal{D}'_{AAut - AP_\omega}(X) \cap \mathcal{D}'_{Q - AP}(X)] \cap \mathcal{D}'_{Q - AP}(X) = \emptyset \);
3. \( \mathcal{D}'_{AAut - AP_\omega}(X) \cap \mathcal{D}'_{Q - AP}(X) = \mathcal{D}'_{AP_\omega}(X) \).
Theorem 3.2. Let $\omega \in I$. It holds $\mathcal{D}^*_e(D_{\omega}) \subseteq \mathcal{D}^*_e(D_{\omega})$.

Theorem 3.3. Let $\omega \in I$. Then
\[ \mathcal{D}^*_e(D_{\omega}) \cap \mathcal{D}^*_e(D_{\omega}) \subseteq \mathcal{D}^*_e(D_{\omega}). \]

Theorem 3.4. The following statements hold:
\begin{enumerate}[i)]  
  \item $\mathcal{D}^*_e(D_{\omega}) \cap \mathcal{D}^*_e(D_{\omega}) = \mathcal{D}^*_e(D_{\omega})$  
  \item $\mathcal{D}^*_e(D_{\omega}) \cap \mathcal{D}^*_e(D_{\omega}) = \mathcal{D}^*_e(D_{\omega})$.
\end{enumerate}

Theorem 3.5. It holds that $\mathcal{D}^*_e(D_{\omega}) \subseteq \mathcal{D}^*_e(D_{\omega})$.

In order to span the investigation on the space of (Q-AP) ultradistributions $\mathcal{D}^*_e(D_{\omega})$, following the approach of B. Basit and H. G"uenzler [1] (see also [24]), now we will focus to the case when $A = Q - AP(R : X)$. Then $A$ is closed under the uniform convergence and addition, and we have $A \subseteq \mathcal{D}^*_e(X)$ ( [23]).

Theorem 3.6. Let $(M_p)$ satisfies the conditions (M.1), (M.2) and (M.3)' and let $T \in \mathcal{D}^*_e(X)$. Then the following holds:
\begin{enumerate}[i)]  
  \item Suppose that there exist an ultradifferential operator $P(D) = \sum_{p=0}^{\infty} a_p D^p$ of class $(M_p)$, resp., of class $(M_p)$ and $f, g \in \mathcal{D}^*_e(D_{\omega})$ such that $T = P(D)f + g$. Then $T \in \mathcal{D}^*_e(D_{\omega})$.
  \item If $T \in \mathcal{D}^*_e(D_{\omega})$ then there exists $h > 0$ such that for each compact set $K \subseteq \mathbb{R}$ we have $T * \varphi \in Q - AP(R : X)$, for all $\varphi \in \mathcal{D}^*_e(D_{\omega})$.
  \item Suppose that $T \in \mathcal{D}^*_e(D_{\omega})$ and there exists $h > 0$ such that for each compact set $K \subseteq \mathbb{R}$ we have $T * \varphi \in Q - AP(R : X)$, for all $\varphi \in \mathcal{D}^*_e(D_{\omega})$. If $(M_p)$ additionally satisfies (M.3), then there exist $l > 0$ and two elements $f, g \in Q - AP(R : X)$ such that $T = P(D)f + g$.
\end{enumerate}

As a consequence of the Theorem 3.6, we immediately have the following:

Corollary 3.1. Let $(M_p)$ satisfies the conditions (M.1), (M.2) and (M.3)' and let $T \in \mathcal{D}^*_e(X)$. Consider the following statements:
\begin{enumerate}[i)]  
  \item $T \in \mathcal{D}^*_e(D_{\omega})$.
  \item There exist a number $l > 0$, resp., a positive increasing sequence $(r_p)$ and a two functions $f, g \in Q - AP(R : X)$ such that $T = P_l(D)f + g$, resp., $T = P_{r_p}(D)f + g$.
  \item There exist an ultradifferential operator $P(D) = \sum_{p=0}^{\infty} a_p D^p$ of class $(M_p)$, resp., $(M_p)$ and two functions $f, g \in Q - AP(R : X)$ such that $T = P(D)f + g$.
  \item There exists $h > 0$ such that for each compact set $K \subseteq \mathbb{R}$, resp., for each $h > 0$ and for each compact set $K \subseteq \mathbb{R}$, we have $T * \varphi \in Q - AP(R : X)$, for all $\varphi \in \mathcal{D}^*_e(D_{\omega})$.
Then we have $ii) \Rightarrow iii) \Rightarrow i) \Leftrightarrow iv)$. Furthermore, if $(M_p)$ additionally satisfies the condition $(M.3)$, then the assertions $(i) - (iv)$ are equivalent for the Beurling class.

Similarly, like in distribution case we have the following theorem:

**Theorem 3.7.** Let $T \in \mathcal{D}_{L_1}^\ast(X)$. The then following holds:

i) Let $T \in \mathcal{D}_{Q-AP}^\ast(X)$, resp., $T \in \mathcal{D}_{SPQ-AP}^\ast(X)$. Then $eT$ is a $(Q-AP)$ ultradistribution, resp., $(SPQ-AP)$ ultradistribution, for any $c \in C$;

ii) If $(T_n)$ is a sequence in $\mathcal{D}_{Q-AP}^\ast(X)$, resp., $\mathcal{D}_{SPQ-AP}^\ast(X)$ and $T_n \to T$ uniformly in $\mathcal{D}_{Q-AP}^\ast(X)$, resp., $\mathcal{D}_{SPQ-AP}^\ast(X)$, then $T \in \mathcal{D}_{Q-AP}^\ast(X)$, resp., $T \in \mathcal{D}_{SPQ-AP}^\ast(X)$;

iii) Any translation $T_h = \langle T, \varphi(\cdot - h) \rangle$ of $T \in \mathcal{D}_{Q-AP}^\ast(X)$ ($T \in \mathcal{D}_{SPQ-AP}^\ast(X)$) is again in $\mathcal{D}_{Q-AP}^\ast(X)$ ($\mathcal{D}_{SPQ-AP}^\ast(X)$).

Let we introduce the following space

$$\mathcal{E}_{Q-AP}^\ast = \{ \varphi \in \mathcal{E}^\ast(X) : \varphi^{(i)} \in Q - AP(\mathbb{R} : X) \text{ for all } i \in \mathbb{N}_0 \}.$$ 

We have that $\mathcal{E}_{Q-AP}^\ast(X) \subseteq \mathcal{D}_{L_1}^\ast(X)$, $\mathcal{E}_{Q-AP}^\ast = \mathcal{E}^\ast(X) \cap Q - AP(\mathbb{R} : X)$ and $\mathcal{E}_{Q-AP}^\ast \ast L^1(\mathbb{R}) \subseteq \mathcal{E}_{Q-AP}^\ast(X)$. Furthermore, the space of $\mathcal{E}_{Q-AP}^\ast(X)$ is the space of all mappings $f$ from $\mathcal{E}^\ast(X)$ for which $f \ast \varphi \in Q - AP(\mathbb{R} : X)$ for all $\varphi \in \mathcal{D}^\ast$.

The proof of the following theorem is the same like [24, Lemma 1].

**Theorem 3.8.** Let $(M_p)$ satisfy the conditions $(M.1)$, $(M.2)$ and $(M.3)'$ and let $T \in \mathcal{D}_{L_1}^\ast(X)$. We have the following statements are equivalent:

i) $T \in \mathcal{D}_{Q-AP}^\ast(X)$.

ii) There exists a sequence $(T_n)$ in $\mathcal{E}_{Q-AP}^\ast(X)$ such that $\lim_{n \to \infty} T_n = T$ in the topology of $\mathcal{D}_{L_1}^\ast(X)$.

Likewise in distribution case, we have the following theorem:

**Theorem 3.9.** Let $(M_p)$ satisfy the conditions $(M.1)$, $(M.2)$ and $(M.3)'$ and let $T \in \mathcal{D}_{L_1}^\ast(X)$. We have the following statements:

i) $T \in \mathcal{D}_{Q-AP}^\ast(X)$.

ii) There exist an element $S \in \mathcal{D}_{L_1}^\ast(X)$, a number $l > 0$ in the Beurling case (a sequence $(r_p) \in \mathbb{R}$ in the Roumieu case), and bounded functions $f, g \in Q - AP(\mathbb{R} : X)$ such that $S = P_l(D)f + g$, resp. $S = P_{r_p}(D)f + g$, and $S = T$ on $[0, \infty)$.

iii) There exist a number $l > 0$, resp. a sequence $(r_p) \in \mathbb{R}$, and bounded functions $f, g \in Q - AP(\mathbb{R} : X)$ such that $T = P_l(D)f + g$, resp. $T = P_{r_p}(D)f + g$, on $[0, \infty)$.

iv) There exist an ultradifferential operator $P(D) = \sum_{p=0}^\infty a_p D^p$ of $*$-class and bounded functions $f_1, f_2 \in Q - AP(\mathbb{R} : X)$ such that $T = P(D)f_1 + f_2$ on $[0, \infty)$.

v) There exist an element $S \in \mathcal{D}_{L_1}^\ast(X)$, an ultradifferential operator $P(D) = \sum_{p=0}^\infty a_p D^p$ of $*$-class and bounded functions $f_1, f_2 \in Q - AP(\mathbb{R} : X)$ such that $S = P(D)f_1 + f_2$ and $S = T$ on $[0, \infty)$. 

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(vi) \( T \in D^{\ast}_{Q-AP}(X) \) and there exists an element \( S \in D^{\ast}_{L^{1}}(M_{p}:X) \) such that \( S = T \) on \([0, \infty)\) and \( S \ast \phi \in Q - AP(\mathbb{R}:X), \phi \in D^{\ast} \).

Then we have ii) \( \Rightarrow \) iii) \( \Rightarrow \) iv) \( \Rightarrow \) v) \( \Rightarrow \) vi) \( \Rightarrow \) i). If \((M_{p})\) additionally satisfies the condition \((M.3)\), then the upper statements are equivalent for the Beurling case.

**Proof.** The implications ii) \( \Rightarrow \) iii) \( \Rightarrow \) iv) \( \Rightarrow \) v) are trivial. We are going to prove that v) \( \Rightarrow \) vi). Let \( S := P(D)g_{1} + g_{2} \) is a (Q-AP) ultradistribution of Beurling class, for \( f_{1}, f_{2}, g_{1}, g_{2} \in Q - AP(\mathbb{R}:X) \) We need to show is

\[
T_{2} := P(D)(f_{1} - g_{1}) + (f_{2} - g_{2}) \in D^{(M_{p})}(X), \text{ i.e.,}
\lim_{h \to +\infty} \left\langle P(D)(f_{1} - g_{1}) + (f_{2} - g_{2}), \phi(\cdot - h) \right\rangle = 0, \quad \phi \in D^{(M_{p})}.
\]

Towards this end, assume that \(-\infty < a < b < +\infty\) and \( \text{supp}(\phi) \subseteq [a, b] \). Let \( \epsilon > 0 \) be given. Then there exist a sufficiently large finite number \( h_{0}(\epsilon, h) > 0 \) and a sufficiently large finite number \( c_{\phi} > 0 \) independent of \( \epsilon \), such that, for every \( |t| \geq h_{0}(\epsilon, h) \), we have the following (cf. also the proof of [24, Theorem 1]):

\[
\left\| \left\langle P(D)(f_{1} - g_{1}) + (f_{2} - g_{2}), \phi(\cdot - h) \right\rangle \right\| \leq \sum_{p=0}^{\infty} (-1)^{p} a_{p} \int_{-\infty}^{\infty} \left( f_{1}(t) - g_{1}(t) \right) \phi^{(p)}(t) \phi(t - h) dt + \int_{-\infty}^{\infty} (f_{2}(t) - g_{2}(t)) \phi(t - h) dt \leq \epsilon c_{\phi}.
\]

This yields vi). By Theorem 3.6 i), we have that vi) \( \Rightarrow \) i). If \((M_{p})\) additionally satisfies the condition \((M.3)\), in Beurling case, holds i) \( \Rightarrow \) ii), so all the upper statements are equivalent in Beurling case. \( \square \)

4. Applications

Let \( n \in \mathbb{N} \), and let \( A = [a_{ij}]_{1 \leq i, j \leq n} \) be a given complex matrix such that \( \sigma(A) \subseteq \{z \in \mathbb{C} : \Re z < 0\} \). In this section, we analyze the existence of (Q-AP) (ultra) distribution solutions of the equation

\[
T' = AT + F, \quad T \in D'(X^{n}) \text{ on } [0, \infty)
\]

and the equation

\[
T' = AT + F, \quad T \in D^{\ast}(X^{n}) \text{ on } [0, \infty),
\]

where \( F \) is a (Q-AP) \( X^{n} \)-valued distribution in (4.1) and \( F \) is a (Q-AP) \( X^{n} \)-valued ultradistribution of \( \ast \)-class in (4.2). By a solution of (4.1), resp. (4.2), we mean
any distribution \( T \in \mathcal{D}'(X^n) \), resp. \( \mathcal{D}^*(X^n) \), such that (4.1), resp. (4.2), holds in distributional, resp. ultradistributional, sense on \([0, \infty)\).

Now, we can state the following theorem, as a consequence [25, Theorem 4.1]:

**Theorem 4.1.**

(i) Let \( F = [F_1, F_2, \ldots, F_n]^T \in \mathcal{D}'_{Q-AP}(X^n) \). Then there exists a solution \( T = [T_1, T_2, \ldots, T_n]^T \in \mathcal{D}'_{Q-AP}(X^n) \) of (4.1). Furthermore, any distributional solution \( T \) of (4.1) belongs to the space \( \mathcal{D}'_{Q-AP}(X^n) \).

(ii) Let \( (M_p) \) satisfy the conditions (M.1), (M.2) and (M.3)', and let \( F = [F_1, F_2, \ldots, F_n]^T \in \mathcal{D}^*(X^n) \) be such that, for every \( i \in \mathbb{N}_n \), there exist an ultradifferential operator \( P_i(D) = \sum_{p=0}^{\infty} a_{i,p} D^p \) of \(*\)-class and bounded functions \( f_{1,i}, f_{2,i} \in \mathcal{Q}_{AP}(\mathbb{R}:X) \) such that \( F_i = P_i(D) f_{1,i} + f_{2,i} \), on \([0, \infty)\). Then there exist ultradifferential operators \( P_{ij}(D) \) of \(*\)-class, bounded functions \( h_{ij} \in \mathcal{Q}_{AP}(\mathbb{R}:X) \) and bounded functions \( h_{2,i} \in \mathcal{Q}_{AP}(\mathbb{R}:X) \) (1 \( \leq \) \( i \leq \) \( n \)), such that \( T = [T_1, T_2, \ldots, T_n]^T \in \mathcal{D}'_{Q-AP}(X^n) \) is a solution of (4.2), where \( T_i = \sum_{j=1}^{n} P_{ij}(D) h_{ij} + h_{2,i} \), for \( i \in \mathbb{N}_n \). Furthermore, any ultradistributional solution \( T \) of \(*\)-class to the equation (4.2) has such a form.

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**REFERENCES**


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COMPACTNESS IN SINGULAR CARDINALS REVISITED

SAHARON SHELAH

ABSTRACT. This is the second combinatorial proof of the compactness theorem for singular from 1977. In fact it gives a somewhat stronger theorem.

1. INTRODUCTION

For a long time I have been interested in compactness in singular cardinals; i.e., whether if something occurs for “many” subsets of a singular \( \lambda \) of cardinality \(< \lambda \), it occurs for \( \lambda \). For the positive side in the seventies we have

**Theorem 1.1.** Let \( \lambda \) be a singular cardinal, \( \chi^* < \lambda \). Let \( \mathcal{U} \) be a set, \( F \) a family of pairs \((A, B)\) of subsets of \( \mathcal{U} \), instead of \((A, B) \in F\) we may write \( A/B \in F \) (formal quotient) or \( A/B \) is \( F \)-free. Assume further that \( F \) is a nice freeness notion meaning it satisfies axioms II, III, IV, VI, VII from 1.1 below. Let \( A^*, B^* \subseteq \mathcal{U} \) with \( |B^*| = \lambda \). Then \( B^*/A^* \in F \) is free in a weak sense, that is: there is an increasing continuous sequence \( \langle A_\alpha : \alpha < \delta \rangle \) of subsets of \( B^* \) of cardinality \(< \lambda \) such that \( A_0 = \emptyset \), \( \bigcup A_\alpha = A^* \) and \( A_{i+1}/A_i \cup A \) is \( F \)-free for \( i < \lambda \) when \( (\text{see Definition 1.2 below}) : (\ast)_0 \) for the \( \mathcal{D}_{\chi^*}(B^*) \)-majority of \( B \in [B^*]^{<\lambda} \) we have \( B/A^* \in F \) or just

\begin{align*}
(\ast)_1 & \text{ the set } \{ \mu < \lambda : \{ B \in [B^*]^{\mu} : B/A^* \in F \} \in \mathcal{E}_{\mu}^{(B^*)} \} \text{ contains a club of } \lambda, \\
(\ast)_2 & \text{ for some set } C \text{ of cardinals } < \lambda, \text{ unbounded in } \lambda \text{ and closed (meaningful only if } \text{ cf}(\lambda) > \aleph_0) \text{, for every } \mu \in C, \text{ for an } \mathcal{E}_{\mu}^{(B^*)} \text{- positive set of } B \in [B^*]^{\mu} \text{ we have } B/A^* \in F.
\end{align*}

Where

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Research supported by the United States-Israel Binational Science Foundation. In a mimeographed from this was included in May 1977, and a lecture on it was given in Berlin 1977. References like \[7, \text{ Th0.2} = \text{ Ly5}\] means the label of Th.0.2 is y5. The reader should note that the version in my website is usually more updated than the one in the mathematical archive. Publication 266.
**Definition 1.1.** For a set $\mathcal{U}$ and $F \subseteq \{(A,B) : A,B \subseteq \mathcal{U}\}$ but we may write $B/A$ instead $(A,B)$, we say, $F$ is a $\chi$-nice freeness notion if $F$ satisfies:

- **Ax.II** $B/A \in F \iff A \cup B/A \in F$
- **Ax.III** if $A \subseteq B \subseteq C, B/A \in F$ and $C/B \in F$ then $C/A \in F$,
- **Ax.IV** if $\langle A_i : i \leq \theta \rangle$ is increasing continuous, $\theta = cf(\theta), A_{i+1}/A_i \in F$ then $A_0/A_0 \in F$,
- **Ax.VI** if $A/B \in F$ then for the $\mathcal{D}_\chi$-majority of $A' \subseteq A$ we have, $A'/B \in F$ (see below),
- **Ax.VII** if $A/B \in F$ then for the $\mathcal{D}_\chi$-majority of $A' \subseteq A$ we have, $A/B \cup A' \in F$.

**Definition 1.2.**

1) Let $\mathcal{D}$ be a function giving for any set $B^*$ a filter $\mathcal{D}(B^*)$ on $\mathcal{P}(B^*)$ (or on $[B^*]^\mu$).

Then to say “for the $\mathcal{D}$-majority of $B \subseteq B^*$ (or $B \in [B^*]^\mu$) we have $\varphi(B)$” means $\{B \subseteq B^* : \varphi(B)\} \in \mathcal{D}(B^*)$ (or $\{B \in [B^*]^\mu : \neg \varphi(B)\} = \emptyset$ mod $\mathcal{D}(B^*)$).

2) Let $\mathcal{D}_\mu(B^*)$ be the family of $Y \subseteq \mathcal{P}(B^*)$ such that for some algebra $M$ with universe $B^*$ and $\leq \mu$ functions,

$$ Y \supseteq S_M = \{B \subseteq B^* : B \neq \emptyset \text{ is closed under the functions of } M\}.$$

2A) Let $\mathcal{D}_{=\mu}(B^*)$ be defined similarly considering only $B$’s of cardinality $\leq \mu$.

3) $E_\mu^+(B^*)$ where $\mu \leq \kappa ^+$ is the collection of all $Y \subseteq [B^*]^\kappa$ such that: for some $\chi, x$ satisfying $\{B^*, x\} \in H(\chi)$, if $M = \langle M_i : i < \mu \rangle$ is an increasing continuous sequence of elementary submodels of $H(\chi), \in \rangle$ such that $x \in M_0, \kappa + 1 \subseteq M_0, ||M_i|| = \kappa$ and $i < \mu \Rightarrow M \upharpoonright (i+1) \subseteq M_{i+1}$, then

(a) if $\mu \leq \kappa$ then $\bigcup_{i<\mu} M_i \cap B^* \in Y$

(b) if $\mu = \kappa ^+$ then for some club $C$ of $\mu^+$ we have $i \in C \Rightarrow M_i \cap B^* \in Y$.

On $\mathcal{D}_\mu$ see Kueker [6], and on $E_\mu^+$ see [9] repeated in §2 below, note that in [9] the axioms are phrased with elementary submodels rather then saying “majority”.

The theorem was proved in [9] but with two extra axioms, however it included the full case for varieties (i.e., including the non-Schreier ones). Later, the author eliminated those two extra axioms: Ax.V and Ax.I. Now Ax.V was used in one point only in [9, §1], and I eliminated it early (as presented in [1]). Axiom I is more interesting: it say that if $A' \subseteq A$ and $A/B$ free then $A'/B$ is $F$-free”; this is like “every subgroup of a free group if free; (this was shown not to be necessary for varieties already in [9]).

In 77 Fleissner has asked for a simpler “combinatorial” proof and we find such proof circulateding it in mimeographed notes [10]. In May 77, and lecture on it in Berlin (summer 77 giving the full details only for the case close to Abelian groups). This proof eliminates the two extra axioms (as its assumptions holds by [9, Lemma 3.4,p.349], see §2 below).

Continuing this Hodges do [5] which contain a compactness result and new important applications. I have thought he just represent the theorem but looking at
it lately it seems to me this is not exactly so; the main point in the proof appears
but the frame is different so it is relative. This exemplifies the old maxim “if you
want things done in the way you want it, you have to do them yourself”.

Anyhow below in §1,§2 we repeat the mimeographed notes. Note that §2 repeats
[9, 3.4] needed for deducing 1.1. Restricted to the needed case; note 3.3 give
hypothesis I (the non $E_{\lambda_1}^{\lambda_i^+}$ -non freeness is $(*)_2$ of 1.1 where hypothesis II is a weak
form of Ax VII.

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after many years.

2. A COMPACTNESS THEOREM FOR SINGULAR

Here we somewhat improve and simplify the proof of [9] (and [1]). It may be
considered an answer to question B2 of Fleissner [4].

**Theorem 2.1.** Assume

(a) $\lambda$ is a singular cardinal, $\lambda_i(i < \kappa)$ an increasing and continuous sequence of
cardinals (we let $\lambda(i) = \lambda_i$) and

$$\lambda_0 = 0, \kappa = \text{cf}(\lambda), \kappa \leq \lambda_1, \lambda = \sum_{i < \kappa} \lambda_i.$$

(b) Let $S_i = \{A \subseteq \lambda : |A| \leq \lambda_i\}$ and $S_i' = S_i \cup \{\emptyset\}$

(c) $F$ is a family$^1$ of pairs $(A, B), \lambda \supseteq A \supseteq B$; we may write “$A/B$ belong to $F$”

(d) hypothesis I: for each $i, i < \kappa, i$ a successor, there is a function $g_i$, two-place,
from $S_i'$ to $S_i'$, such that: if $A_1 \subseteq A_2$ are from $S_i', A_1 \in \{\emptyset\} \cup \text{Range}(g_i)$, then
$A_2 \subseteq g_i(A_1, A_2)$ and $[g_i(A_1, A_2)/A_1] \in F$

(e) hypothesis II: if $i < \kappa, A, B \in S_{i+1}'$, $A \subseteq B$ and $B/A \in F$ and $B \in \text{Rang}(g_{i+1})$,
then player II has a winning strategy in the following game $G_{m_i}$.
In the $n$-th move ($n < \omega$) player I choose $A_n \in S_i$, such that $B_{n-1} \subseteq A_n$, and then
player II choose $B_n$, such that $A_n \subseteq B_n \in S_i$ (where we stipulate $B_{-1} = \emptyset$).
Player II wins in the play if $(B \cup \bigcup_{n < \omega} B_n, A \cup \bigcup_{n < \omega} B_n) \in F$ (for $i = 0$ this is an
empty demand as $S_i' = \{\emptyset\}$).

Then we can find an increasing and continuous chain $A_\alpha(\alpha < \omega \kappa)$, such that
$A_0 = 0, \lambda = \bigcup_{\alpha} A_\alpha$ and $A_{\alpha+1}/A_\alpha \in F$ for each $\alpha$.

**Proof.** Let in Hypothesis II the winning strategy of player II in the game $G_{m_i}$ be
given by the functions $h_i^n(A_0, \ldots, A_n; A, B)$. We define by induction on $i < \omega$ sets
$A_i^n, B_i^n$ (for $i < \kappa$) such that:

1. $A_i^n(i < \kappa)$ is increasing and continuous in $i$ and $A_i^n, B_i^n \in S_i$
2. $A_i^n \subseteq B_i^n \subseteq A_i^{n+1}$

$^1$Note that none of the axioms of 1.1 is assumed.
In the construction proving the Theorem we can continue (2) usually the choice of the hypothesis II: if \( \omega < \lambda, B_i^0 = g_i(0, \lambda_i); \) clearly condition (1) holds, (2) and (4) say nothing and condition (3) holds by Hypothesis I. 

For \( n + 1 \) assuming that for \( n \) we have defined. 

Let 

\[
C^n_i = \bigcup_{m<n} h^n_{i = m-1}(A^m_{i+1}, A^m_{i+2}, \ldots, A^n; B^m_{i+1}, B^m_{i+1}) \cup B^n_i
\]

clearly \( |C^n_i| = \lambda_i \), hence we can let \( C^n_i = \{\alpha b^n_i : \alpha < \lambda_i\} \).

Now we define \( A^n_{i+1} = \{\alpha b^n_j : j < \kappa, \alpha < \min\{\lambda_i, \lambda_j\}\} \).

Clearly condition (4) and the relevant parts of conditions (1) and (2) hold. We have to choose \( B^n_{i+1} \) such that 

\[
A^n_{i+1} \subseteq B^n_{i+1} \quad \text{and} \quad |B^n_{i+1}| = \lambda_i, \quad \text{and i successor} \Rightarrow B^n_{i+1}/B^n_i \in F.
\]

So we let \( B^n_{i+1} = g_i(B^n_i, A^n_{i+1}) \) except that \( B^n_{0+1} = 0 \). By Hypothesis I this is O.K.

Now we can prove the conclusion of the theorem.

We let \( D_{\omega+1} = (B^1_{i+1} \cap \bigcup_{m<\omega} A^n_i) \cup \bigcup_{j<i, m<\omega} A^m_j \) for \( i < \kappa \). Clearly \( D_0 = 0 \), (in fact \( A^n_i, B^n_i \) are \( 0 \) for \( i = 0 \)); \( \lambda = \bigcup_{i<\omega} D_i \) as \( \lambda_i = A^n_i \subseteq D_\omega(i+1) \subseteq \lambda \). The sequence is increasing and continuous.

[that is e.g., if \( \delta = \omega i + \omega \) so \( \delta = \omega(i+1) + 0 \) then \( D_\delta \subseteq \bigcup_{\alpha < \delta} D_\alpha \) as \( B^{-1}_{i+1} = 0 \), so \( D_\delta = \bigcup_{j<i+1, m<\omega} A^m_j \cup \bigcup_{j<i, m<\omega} A^m_j \) but \( A^m_i \subseteq A^m_{i+1} \subseteq B^m_{i+1} \).

so \( D_\delta \subseteq \bigcup_{k} \bigg[ \bigcup_{j<i, m<\omega} A^m_j \cup (B^k_{i+1} \cup \bigcup_{m<\omega} A^m_i) = \bigcup_{D_{\omega+1} \subseteq \bigcup_{\alpha < \delta}} \bigg] \).

Now \( D_{\omega+1} \subseteq D_{\omega+1} \in F \) as \( B^k_{i+1} \subseteq B^k_{i+1} \subseteq F \) by condition (3), and then use condition (4)) and the choice of the \( h^n_i - s' \) [that is, player II wins the play 

\[
(A^n_{i+1}, h^n_i \subseteq (A^{k+1}_{i+1}, A^{k+2}_{i+1} \ldots, A^{k+\ell}_{i+1}, B^k_{i+1})): \ell < \omega)
\]

of the game \( G_{\omega+i}(B^n_{i+1}, B^n_{i+1}) \).

\( \square \)

Remark 2.1. 1) In the context of [8], [1] Hypothesis I holds quite straightforwardly whereas Hypothesis II is proved separately, see [9, Lemma 3.4 p. 344].

2) Usually the choice of the \( \lambda_i \)'s is not important, and then Hypothesis I, Hypothesis II should speak on \( \mu < \lambda, \mu < \mu' < \lambda \).

3) In the construction proving the Theorem we can continue \( \chi < \lambda_1 \) steps instead of \( \omega \) steps. We succeed if: in Hypothesis II the game has length \( \chi \) and we add to hypothesis II: if \( A_i/A_0 \in F \) for \( i < \chi, A_i \) increasing continuous then \( \bigcup_{i<\chi} A_i/A_0 \in F \).
An example is: G is a group with universe \( \lambda \) and

\[ F = \{(A, B) : \text{Ext}(A/B, C^+) = \emptyset\} \]

where \( A \subseteq B \) are subgroup of \( G \), \( \text{cf}(\lambda) < \chi < \lambda \), \( \chi \) measurable \( (C^+ \text{ a fixed group of cardinality } < \chi) \) and e.g. G.C.H. (see below).

4) We can improve a little Eklof’s results on compactness [3] where “\( A \) free” is replace by “\( \text{Ext}(A, \mathbb{Z}) = 0 \)”.

Note that in his proofs \( \diamond_S \) can be replaced by “\( S \) not small” e.g. (see [2]), and instead “\( \diamond_S \) for stationary \( S \)” by the above “\( S \) not small for all stationary \( S \) such that \( (\forall \delta \in S) \text{cf}(\delta) = \aleph_0 \)” suffice but if \( \sup(S) = \lambda^+, \lambda^{\aleph_0} = \lambda, 2^\lambda = \lambda^+ \), this holds. So we can get compactness for \( \beth_{\alpha+\omega} \) assuming G.C.H.

4A) Hypothesis I can be rephrased similary to Hypothesis II, as the existence of a winning strategy (to player II) in appropriate game.

5) For the Whithead problem we need only “any \( \lambda \)-free abelian group is \( \lambda^+ \)-free” for singular \( \lambda \). So suppose \( G \) is a \( \lambda -\)free group with universe \( \lambda \) and \( F = \{(A, B) : A/B \text{ is free}\} \). There we do not need Hypothesis I, and can represent the proof somewhat differently.

In the construction we choose pure subgroups \( A^n_i, B^n_i \) and choose a free basis \( I^n_i \) of \( A^n_i \) and demand satisfying

\( (a) \) (1) + (2)

\( (b) \) for \( m < n \), \( A^m_{i+1} \cap B^n_i \) is generated by a subset of \( I^m_{i+1} \)

\( (c) \) for each \( m < n \) and integer \( a \),

\[(\forall x \in B^n_i \cap A^m_{i+1})[\exists y \in A^{m+1}_{i+1}] (ax + x \in A^m_{i+1}) \Rightarrow (\exists y \in A^{m+1}_{i+1} \cap B^n_i) ay + x \in A^m_{i+1}] \]

By \( (c) \) we shall get \( A^m_{i+1} / \bigcup_{m<n} B^n_i \) hence it is known (Hill) that

\[ \bigcup_{m<n} A^m_{i+1} / \bigcup_{m<n} B^n_i = \bigcup_{n} A^n_i \]

is free thus finishing.

3. ON THE HYPOTHESIS

Context 3.1. \( \mathcal{U}, F \) is as in Definition 1.1.

Notation 3.2. 0) \( A, B, C, D \) denote subsets of \( \mathcal{U} \).

1) \( \mathcal{S}_\kappa(A) = \{B \subseteq A : |B| < \kappa\} \).

2) \( A/B \) is free mean \( (A, B) \in F \).

3) \( A, B, D \) denote subsets of \( \mathcal{U} \).

4) \( \mathcal{M} = (\mathcal{H}(\chi), \subseteq, <^*_\chi) \) where \( \chi \) is large enough such that \( \mathcal{P}(\mathcal{U}) \subseteq \mathcal{H}(\chi) \) and \( <^*_\chi \) a well ordering of \( \mathcal{M} \). We say \( \mathcal{M}^\chi \) is a \( \kappa \)-expansion of \( \mathcal{M} \) if we expand \( \mathcal{M} \) by \( \leq \kappa \) additional relations and functions.

5) \( \mathcal{S}_\kappa^{\text{ub}}(A) \) is the following filter or \( \mathcal{S}_\kappa(A) : Y \in \mathcal{S}_\kappa^{\text{ub}}(A) \) iff \( Y \supseteq Y_C = Y_C(A) \) for some \( Y \subseteq A, C \in \mathcal{S}_\kappa(A) \) where \( Y_C = \{B \in \mathcal{S}_\kappa(A) : C \subseteq B\} \) we call \( Y_C[A] \) a generator.
Remark 3.1. Obvious monotonicity results hold.

Definition 3.2. 1) For every $\mu \leq \kappa < \lambda, C \in \mathcal{S}_\kappa(A), A \subseteq \mathcal{U}$ such that $|A| = \lambda$, and $B \subseteq \mathcal{U}$ and filter $\mathcal{E}$ over $\mathcal{S}_\kappa(A)$, we define the rank $R(C, \mathcal{E}) R^\mu(C, \mathcal{E}; A/B)$ as an ordinal or $\infty$, so that

(a) $R(C, \mathcal{E}) \geq \alpha + 1$ iff $C/B$ is free and $\{ D \in \mathcal{S}_\kappa(A) : C \subseteq D$ and $D/C \cup B$ is free and $R(D, \mathcal{E}) \geq \alpha \} \neq \emptyset \mod \mathcal{E}$

(b) $R(C, \mathcal{E}) \geq \delta(\delta = 0$ or $\delta$ limit) iff $C/B$ is free and $\alpha < \delta$ implies $R^\mu(C, \mathcal{E}) \geq \alpha$.

2) $R(A/B, \mathcal{E}) = \sup \{ R^\mu(C, \mathcal{E}) : C \in \mathcal{S}_\kappa(A) \}.$

3) Writing $R^\mu(C) = R^\mu(C; A/B)$ means $R(C, \mathcal{E}^\mu; A/B)$ and writing $R^\mu_{\text{ub}}(C) = R^\mu_{\text{ub}}(C; A/B)$ means $R(C, \mathcal{E}^\mu_{\text{ub}}; A/B)$. Similarly $R^\mu_{\text{ub}}(A/B)$ means $R(A/B, \mathcal{E}^\mu_{\text{ub}})$ and $R^\mu_{\text{ub}}(A/B)$ means $R(A/B, \mathcal{E}^\mu_{\text{ub}})$.

Remark 3.2. Note that omitting $A/B$ is reasonable because mostly they are clear from the content.

Lemma 3.3. Suppose $\kappa^+ < \lambda, \mu \leq \kappa, A/B$ is not $\mathcal{E}^{\kappa^+}_{\text{ub}}$-non-free and $S_1 \in \mathcal{E}^{\kappa^+}_{\text{ub}}(A)$. Then $R^\mu_{\text{ub}} = \infty,$ moreover for every $S_1 \in \mathcal{E}^{\kappa^+}_{\text{ub}}(A)$ and $\kappa$-expansion $\mathcal{M}^*$ of $\mathcal{M}$ there are $C \in S_2$ and $D \in S_1$ and $N \prec \mathcal{M}^*, \{ A, B \} \in N, ||N|| = \kappa$ such that $D \in N$, $C = D \cap N$ and $R^\mu_{\text{ub}}(C) = \infty.$

Proof. Let $S_1 \supseteq \mathcal{S}_\kappa(\mathcal{M}^*)$ if $C \in \mathcal{S}_\kappa(A), 0 \leq R^\mu_{\text{ub}}(C) < \infty,$ then there is a generator $S(C) \in \mathcal{E}^\mu_{\text{ub}}(A), S(C) = \mathcal{S}^\mu_{\text{ub}}(\mathcal{M}^*),$ such that for $D \in S(C), D/C \cup B$ is not free or $R^\mu_{\text{ub}}(D) < R^\mu_{\text{ub}}(C)$. If $C/B$ is not free or $R^\mu_{\text{ub}}(C) = \infty$, let $\mathcal{M}^c_\kappa$ be any $\kappa$-expansion of $\mathcal{M},$ and let $S_2 = S^\mu_{\text{ub}}(\mathcal{M}^2)$. Let $\mathcal{M}^+$ be a $\kappa$-expansion of $\mathcal{M},$ expanding $\mathcal{M}^*, \mathcal{M}^2$ and having the relations $P, P_2$ where

$P = \{(C, N) : C \in \mathcal{S}_\kappa(A), N \prec \mathcal{M}^c_\kappa, ||N|| < \chi_2\}$

$P_2 = \{ N : N \prec \mathcal{M}^2, ||N|| < \chi_2 \}.$

As

$\{ D \in S^\mu_{\text{ub}}(A) : D/B$ is free $\} \neq \emptyset \mod \mathcal{E}^{\kappa^+}_{\text{ub}}(A)$

and $S_1 \in \mathcal{E}^{\kappa^+}_{\text{ub}}(A)$ and (by 3.1 $\mathcal{S}^{\kappa^+}(A)$); there are $D, \tilde{N}$ such that:

1) $S/B$ is free
2) $D \in S_1$
3) $N_i(i < \kappa^+)$ is an $\mathcal{M}^+$-sequence and $||N_i|| \leq \kappa$, so
4) $D = A \cap \bigcup_{i < \kappa^+} N_i,$ without loss of generality $||N_i|| = \kappa, \kappa \subseteq N_i.$
Let \( A_i^* = D \cap N_i \), so \( A_i^* \subseteq N_{i+1} \) and let \( N = \bigcup_{i<\kappa^+} N_i \). Clearly \( (N_i : i < \kappa^+) \) is also an \( \mathcal{M}^2 \)-sequence hence for each \( \delta < \kappa^+ \), \( (N_i : i < \delta) \) is an \( \mathcal{M}^2 \)-sequence, hence, if \( \kappa \) divides \( \delta \), \( \text{cf}(\delta) = \mu \), then \( A_\delta^* \subseteq S_2 \). If \( C \in N_i, C \subseteq \mathcal{S}_\kappa(A) \), then for every \( j > i, j < \kappa^+ \) there is a model \( N_i^j < \mathcal{M}_C^*, ||N_i^j|| = \kappa, |N_i^j| \) and \( N_i^j \subseteq N_{j+1} \), hence \( N_i^j \subseteq N_{j+1} \).

Hence, for any limit ordinal \( \delta, i < \delta < \kappa^+ \) implies \( N_\delta < \mathcal{M}_C^* \).

Clearly \( (N_i : i < j < \kappa^+, j \text{ limit }) \) is an \( \mathcal{M}^+ \)-sequence, hence it is an \( \mathcal{M}^+ \)-sequence, hence, is \( i < \delta < \kappa^+ \), \( \delta \) is limit, \( \kappa^2 \) divides \( \delta \), \( \text{cf}(\delta) = \mu \), then \( A_\delta^* \subseteq S(C) \). As \( S/B \) is free, by \([9],1.2(7)\) there is a closed unbounded subset of \( \kappa^+, W \), such that for \( i, j \in W, i < j \), \( A_i^* / A_j^* \cup B \) is free and \( A_i^* / B \) is free. We can assume that such \( i \in W \) is divisible by \( \kappa^2 \). Hence, if \( i, j \in W, i < j \), \( \text{cf}(j) = \mu, R_k^{\mu}(A_j^*) < \infty \), then \( R_k^{\mu}(A_j^*) < R_k^{\mu}(A_i^*) < \infty \) (by the definition of \( S(C) \)).

So, if for some \( i \in W, R_k^{\mu}(A_i^*) < \infty, \text{cf}(i_n) = \mu, i_n \in W, i < i_n < i_{n+1} \) then \( R_k^{\mu}(A_i^*) \) is an infinite decreasing sequence of ordinals, a contradiction.

Hence, \( i \in W \) implies \( R_k^{\mu}(A_i^*) = \infty \).

Let \( D = \bigcup_{i < \kappa^+} A_i^* \), and choose \( N \not\subseteq \mathcal{M}^*, D \in N, N \cap \bigcup_{i < \kappa^+} A_i^* = A_\delta^*, \delta \in W, \text{cf}(\delta) = \mu \), and \( C = A_\delta^* \). So we are finished.

\[ \square \]

**Lemma 3.4.** 1) If \( \mu \leq \kappa < \lambda, C \in \mathcal{S}_\kappa(A) \), \( R_k^{\mu}(C) = \infty \), \( S \in \mathcal{E}_k^{\mu}(A) \), then for some \( D \in S, C \subseteq D, R_k^{\mu}(D) = \infty \) and \( D/C \cup B \) is free.

2) The same holds for any filter over \( S_\kappa(A) \).

**Proof.** 1) As \( \mathcal{S}_\kappa(A) \) is a set, for some ordinal \( \alpha_0 < |\mathcal{S}_\kappa(A)|^+ \), for no \( C \in \mathcal{S}_\kappa(A) \) is \( R_k^{\mu}(C) = \alpha_0 \). We can easily prove that \( R_k^{\mu}(C) \geq \alpha_0 \) iff \( R_k^{\mu}(C) = \infty \). Using the definition we get our assertion.

2) The same proof.

\[ \square \]

**References**


ON STRONGLY REGULAR GRAPHS WITH $m_2 = qm_3$ AND $m_3 = qm_2$ FOR $q = 5, 6, 7, 8$

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Dedicated to my mother Jordana Lepović

ABSTRACT. We say that a regular graph $G$ of order $n$ and degree $r \geq 1$ (which is not the complete graph) is strongly regular if there exist non-negative integers $\tau$ and $\theta$ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices $i$ and $j$, and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices $i$ and $j$, where $S_k$ denotes the neighborhood of the vertex $k$. Let $\lambda_1 = r$, $\lambda_2$ and $\lambda_3$ be the distinct eigenvalues of a connected strongly regular graph. Let $m_1 = 1$, $m_2$ and $m_3$ denote the multiplicity of $r$, $\lambda_2$ and $\lambda_3$, respectively. We describe the parameters $n$, $r$, $\tau$ and $\theta$ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 5, 6, 7, 8$.

1. INTRODUCTION

Let $G$ be a simple graph of order $n$ with vertex set $V(G) = \{1, 2, \ldots, n\}$. The spectrum of $G$ consists of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of its (0,1) adjacency matrix $A$ and is denoted by $\sigma(G)$. We say that a regular graph $G$ of order $n$ and degree $r \geq 1$ (which is not the complete graph $K_n$) is strongly regular if there exist non-negative integers $\tau$ and $\theta$ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices $i$ and $j$, and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices $i$ and $j$, where $S_k \subseteq V(G)$ denotes the neighborhood of the vertex $k$. We know that a regular connected graph $G$ is strongly regular if and only if it has exactly three distinct eigenvalues [1] (see also [3]). Let $\lambda_1 = r$, $\lambda_2$ and $\lambda_3$ denote the distinct eigenvalues of a connected strongly regular graph $G$. Let $m_1 = 1$, $m_2$ and $m_3$ denote the multiplicity of $r$, $\lambda_2$ and $\lambda_3$. Further, let $\tau = (n-1) - r$, $\bar{\lambda}_2 = -\lambda_3 - 1$ and $\bar{\lambda}_3 = -\lambda_2 - 1$ denote the distinct eigenvalues of the strongly regular graph $\overline{G}$, where $\overline{G}$ denotes the complement of $G$. Then $\bar{\tau} = n - 2r - 2 + \theta$ and $\bar{\theta} = n - 2r + \tau$, where $\bar{\tau} = \tau(\overline{G})$ and $\bar{\theta} = \theta(\overline{G})$.

Remark 1.1. (i) if $G$ is a disconnected strongly regular graph of degree $r$ then $G = mK_{r+1}$, where $mH$ denotes the $m$-fold union of the graph $H$; (ii) $G$ is a disconnected strongly regular if and only if $\theta = 0$.

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Remark 1.2. (i) a strongly regular graph $G$ of order $n = 4k + 1$ and degree $r = 2k$ with $\tau = k - 1$ and $\theta = k$ is called a conference graph; (ii) a strongly regular graph is a conference graph if and only if $m_2 = m_3$ and (iii) if $m_2 \neq m_3$ then $G$ is an integral\footnote{We say that a connected or disconnected graph $G$ is integral if its spectrum $\sigma(G)$ consists only of integral values.} graph.

We have recently started to investigate strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$, where $q$ is a positive integer [4]. In particular, in the same work we have described the parameters $n$, $r$, $\tau$ and $\theta$ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 2, 3, 4$. We now proceed to establish the parameters of strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 5, 6, 7, 8$, as follows. Firstly,

**Proposition 1.1** (Elzinga [2]). Let $G$ be a connected or disconnected strongly regular graph of order $n$ and degree $r$. Then

$$r^2 - (\tau - \theta + 1)r - (n - 1)\theta = 0. \tag{1.1}$$

**Proposition 1.2** (Elzinga [2]). Let $G$ be a connected strongly regular graph of order $n$ and degree $r$. Then

$$2r + (\tau - \theta)(m_2 + m_3) + \delta(m_2 - m_3) = 0, \tag{1.2}$$

where $\delta = \lambda_2 - \lambda_3$.

**Remark 1.3** (Lepović [4]). Using the same procedure applied in [4] we can establish the parameters $n$, $r$, $\tau$ and $\theta$ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for any fixed value $q \in \mathbb{N}$, as follows. Firstly, let $m_3 = p$, $m_2 = qp$ and $n = (q + 1)p + 1$, where $q \in \mathbb{N}$. Using (1.2) we obtain $r = p(\lceil \lambda_3 \rceil - q\lambda_2)$. Let $\lceil \lambda_3 \rceil - q\lambda_2 = t$, where $t = 1, 2, \ldots, q$. Let $\lambda_2 = k$, where $k$ is a positive integer. Then (i) $\lambda_3 = -(qk + t)$; (ii) $\tau - \theta = -((q - 1)k + t)$; (iii) $\delta = (q + 1)k + t$; (iv) $r = pt$ and (v) $\theta = pt - qk^2 - kt$. Using (ii), (iv) and (v) we can easily see that (1.1) is reduced to

$$(p + 1)t^2 - ((q + 1)p + 1)t + q(q + 1)k^2 + 2qkt = 0. \tag{1.3}$$

Secondly, let $m_2 = p$, $m_3 = qp$ and $n = (q + 1)p + 1$, where $q \in \mathbb{N}$. Using (1.2) we obtain $r = p(q\lceil \lambda_3 \rceil - \lambda_2)$. Let $q\lceil \lambda_3 \rceil - \lambda_2 = t$, where $t = 1, 2, \ldots, q$. Let $\lambda_3 = -k$, where $k$ is a positive integer. Then (i) $\lambda_2 = qk - t$; (ii) $\tau - \theta = (q - 1)k - t$; (iii) $\delta = (q + 1)k - t$; (iv) $r = pt$ and (v) $\theta = pt - qk^2 + kt$. Using (ii), (iv) and (v) we can easily see that (1.1) is reduced to

$$(p + 1)t^2 - ((q + 1)p + 1)t + q(q + 1)k^2 - 2qkt = 0. \tag{1.4}$$

Using (1.3) and (1.4) we can obtain for $t = 1, 2, \ldots, q$ the corresponding classes of strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$, respectively.
2. Main results

**Remark 2.1.** We firstly describe the corresponding classes of connected strongly regular graphs with \( m_2 = qm_3 \) and \( m_3 = qm_2 \) obtained in Propositions 2k + 1 and 2k + 2, respectively, then we prove Theorem k which is related to connected strongly regular graphs with \( m_2 = qm_3 \) or \( m_3 = qm_2 \), where \( q = 5, 6, 7, 8 \) and \( k = 1, 2, 3, 4 \).

**Remark 2.2.** Since \( m_2(G) = m_3(G) \) and \( m_3(G) = m_2(G) \) we note that if \( m_2(G) = qm_3(G) \) then \( m_3(G) = qm_2(G) \).

**Remark 2.3.** In Theorems 2.1, 2.2, 2.3 and 2.4 the complements of strongly regular graphs appear in pairs in \((k^0)\) and \((\overline{k}^0)\) classes, where \( k \) denotes the corresponding number of a class.

**Remark 2.4.** \( \overline{\alpha K_\beta} \) is a strongly regular graph of order \( n = \alpha \beta \) and degree \( r = (\alpha - 1)\beta \) with \( \tau = (\alpha - 2)\beta \) and \( \theta = (\alpha - 1)\beta \). Its eigenvalues are \( \lambda_2 = 0 \) and \( \lambda_3 = -\beta \) with \( m_2 = \alpha(\beta - 1) \) and \( m_3 = \alpha - 1 \).

**Proposition 2.1.** Let \( G \) be a connected strongly regular graph of order \( n \) and degree \( r \) with \( m_2 = 5m_3 \). Then \( G \) belongs to the class \((2^0)\) or \((3^0)\) represented in Theorem 2.1.

**Proof.** Let \( m_3 = p \), \( m_2 = 5p \) and \( n = 6p + 1 \), where \( p \in \mathbb{N} \). Using (1.2) we obtain \( r = p(|\lambda_3| - 5\lambda_2) \). Let \( |\lambda_3| - 5\lambda_2 = t \), where \( t = 1, 2, \ldots, 5 \). Let \( \lambda_2 = k \), where \( k \) is a positive integer. Then according to Remark 1.3 we have (i) \( \lambda_3 = -(5k + t) \); (ii) \( \tau - \theta = -(4k + t) \); (iii) \( \delta = 6k + t \); (iv) \( r = pt \) and (v) \( \theta = pt - 5k^2 - kt \). In this case we can easily see that (1.3) is reduced to

\[
(p + 1)^2 - (6p + 1)t + 30k^2 + 10kt = 0. \tag{2.1}
\]

**Case 1.** \( t = 1 \). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = k \) and \( \lambda_3 = -(5k + 1) \), \( \tau - \theta = -(4k + 1) \), \( \delta = 6k + 1 \), \( r = p \) and \( \theta = p - 5k^2 - k \). Using (2.1) we find that \( 6p - 1 = 2k(3k + 1) \). So we obtain that \( G \) is a strongly regular graph of order \( n = (6p + 1)^2 \) and degree \( r = 2k(3k + 1) \) with \( \tau = k^2 - 3k - 1 \) and \( \theta = k(k + 1) \).

**Case 2.** \( t = 2 \). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = k \) and \( \lambda_3 = -(5k + 2) \), \( \tau - \theta = -(4k + 2) \), \( \delta = 6k + 2 \), \( r = 2p \) and \( \theta = 2p - 5k^2 - 2k \). Using (2.1) we find that \( 4p - 1 = 5k(3k + 2) \), a contradiction because \( 4 \nmid 15k^2 + 10k + 1 \).

**Case 3.** \( t = 3 \). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = k \) and \( \lambda_3 = -(5k + 3) \), \( \tau - \theta = -(4k + 3) \), \( \delta = 6k + 3 \), \( r = 3p \) and \( \theta = 3p - 5k^2 - 3k \). Using (2.1) we find that \( 3p - 2 = 10k(k + 1) \), a contradiction because \( 3 \nmid 10k^2 + 10k + 2 \).

**Case 4.** \( t = 4 \). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = k \) and \( \lambda_3 = -(5k + 4) \), \( \tau - \theta = -(4k + 4) \), \( \delta = 6k + 4 \), \( r = 4p \) and \( \theta = 4p - 5k^2 - 4k \). Using (2.1) we find that \( 4p - 6 = 5k(3k + 4) \), a contradiction because \( 4 \nmid 15k^2 + 20k + 6 \).

**Case 5.** \( t = 5 \). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = k \) and \( \lambda_3 = -(5k + 5) \), \( \tau - \theta = -(4k + 5) \), \( \delta = 6k + 5 \), \( r = 5p \) and \( \theta = 5p - 5k^2 - 5k \). Using
(2.1) we find that \( p = 2(k+1)(3k+2) \). Replacing \( k \) with \( k-1 \) we arrive at \( p = 2k(3k-1) \). So we obtain that \( G \) is a strongly regular graph of order \( n = (6k-1)^2 \) and degree \( r = 10k(3k-1) \) with \( \tau = 25k^2 - 9k - 1 \) and \( \theta = 5k(5k-1) \).

**Proposition 2.2.** Let \( G \) be a connected strongly regular graph of order \( n \) and degree \( r \) with \( m_3 = 5m_2 \). Then \( G \) belongs to the class \((2^0)\) or \((3^0)\) represented in Theorem 2.1.

**Proof.** Let \( m_2 = p, m_3 = 5p \) and \( n = 6p+1 \), where \( p \in \mathbb{N} \). Using (1.2) we obtain \( r = p(5|\lambda_3| - \lambda_2) \). Let \( 5|\lambda_3| - \lambda_2 = t \), where \( t = 1, 2, \ldots, 5 \). Let \( \lambda_3 = -k \), where \( k \) is a positive integer. Then according to Remark 1.3 we have (i) \( \lambda_2 = 5k-t \); (ii) \( \tau - \theta = 4k-t \); (iii) \( \delta = 6k-t \); (iv) \( r = pt \) and (v) \( \theta = pt - 5k^2 + kt \). In this case we can easily see that (1.4) is reduced to

\[(p+1)t^2 - (6p+1)t + 30k^2 - 10kt = 0.\] (2.2)

**Case 1.** \( (t = 1) \). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = 5k-1 \) and \( \lambda_3 = -k \), \( \tau - \theta = 4k-1 \), \( \delta = 6k-1 \), \( r = p \) and \( \theta = p - 5k^2 + k \). Using (2.2) we find that \( p = 2k(3k-1) \). So we obtain that \( G \) is a strongly regular graph of order \( n = (6k-1)^2 \) and degree \( r = 2k(3k-1) \) with \( \tau = k^2 + 3k - 1 \) and \( \theta = k(k-1) \).

**Case 2.** \( (t = 2) \). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = 5k-2 \) and \( \lambda_3 = -k \), \( \tau - \theta = 4k-2 \), \( \delta = 6k-2 \), \( r = 2p \) and \( \theta = 2p - 5k^2 + 2k \). Using (2.2) we find that \( 4p-1 = 5k(3k-2) \), a contradiction because \( 4 \nmid 15k^2 - 10k + 1 \).

**Case 3.** \( (t = 3) \). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = 5k-3 \) and \( \lambda_3 = -k \), \( \tau - \theta = 4k-3 \), \( \delta = 6k-3 \), \( r = 3p \) and \( \theta = 3p - 5k^2 + 3k \). Using (2.2) we find that \( 3p-2 = 10k(k-1) \), a contradiction because \( 3 \nmid 10k^2 - 10k + 2 \).

**Case 4.** \( (t = 4) \). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = 5k-4 \) and \( \lambda_3 = -k \), \( \tau - \theta = 4k-4 \), \( \delta = 6k-4 \), \( r = 4p \) and \( \theta = 4p - 5k^2 + 4k \). Using (2.2) we find that \( 4p-6 = 5k(3k-4) \), a contradiction because \( 4 \nmid 15k^2 - 20k + 6 \).

**Case 5.** \( (t = 5) \). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = 5k-5 \) and \( \lambda_3 = -k \), \( \tau - \theta = 4k-5 \), \( \delta = 6k-5 \), \( r = 5p \) and \( \theta = 5p - 5k^2 + 5k \). Using (2.2) we find that \( p = 2(k-1)(3k-2) \). Replacing \( k \) with \( k+1 \) we arrive at \( p = 2k((3k+1) \). So we obtain that \( G \) is a strongly regular graph of order \( n = (6k+1)^2 \) and degree \( r = 10k(3k+1) \) with \( \tau = 25k^2 + 9k - 1 \) and \( \theta = 5k(5k+1) \).

**Remark 2.5.** We note that \( 5K_5 \) is a strongly regular graph with \( m_2 = 5m_3 \). It is obtained from the class Theorem 2.1 \((2^0)\) for \( k = 1 \).

**Theorem 2.1.** Let \( G \) be a connected strongly regular graph of order \( n \) and degree \( r \) with \( m_2 = 5m_3 \) or \( m_3 = 5m_2 \). Then \( G \) is one of the following strongly regular graphs:

\( (1^0) \) \( G \) is the strongly regular graph \( 5K_5 \) of order \( n = 25 \) and degree \( r = 20 \) with \( \tau = 15 \) and \( \theta = 20 \). Its eigenvalues are \( \lambda_2 = 0 \) and \( \lambda_3 = -5 \) with \( m_2 = 20 \) and \( m_3 = 4 \);
(20) G is a strongly regular graph of order \( n = (6k-1)^2 \) and degree \( r = 2k(3k-1) \) with \( \tau = k^2 + 3k - 1 \) and \( \theta = k(k-1) \), where \( k \geq 2 \). Its eigenvalues are \( \lambda_2 = 5k - 1 \) and \( \lambda_3 = -k \) with \( m_2 = 2k(3k-1) \) and \( m_3 = 10k(3k-1) \);

(21) G is a strongly regular graph of order \( n = (6k-1)^2 \) and degree \( r = 10k(3k-1) \) with \( \tau = 25k^2 - 9k - 1 \) and \( \theta = 5k(5k-1) \), where \( k \geq 2 \). Its eigenvalues are \( \lambda_2 = k - 1 \) and \( \lambda_3 = -5k \) with \( m_2 = 10k(3k-1) \) and \( m_3 = 2k(3k-1) \);

(30) G is a strongly regular graph of order \( n = (6k+1)^2 \) and degree \( r = 2k(3k+1) \) with \( \tau = k^2 - 3k - 1 \) and \( \theta = k(k+1) \), where \( k \geq 4 \). Its eigenvalues are \( \lambda_2 = k \) and \( \lambda_3 = -(5k+1) \) with \( m_2 = 10k(3k+1) \) and \( m_3 = 2k(3k+1) \);

(31) G is a strongly regular graph of order \( n = (6k+1)^2 \) and degree \( r = 10k(3k+1) \) with \( \tau = 25k^2 + 9k - 1 \) and \( \theta = 5k(5k+1) \), where \( k \geq 4 \). Its eigenvalues are \( \lambda_2 = 5k \) and \( \lambda_3 = -(k+1) \) with \( m_2 = 2k(3k+1) \) and \( m_3 = 10k(3k+1) \).

Proof. Firstly, according to Remark 2.4 we have \( \alpha(\beta - 1) = 5(\alpha - 1) \), from which we find that \( \alpha = 5 \), \( \beta = 5 \). In view of this we obtain the strongly regular graph represented in Theorem 2.1 (10). Next, according to Proposition 2.1 it turns out that \( G \) belongs to the class (20) or (30) if \( m_2 = 5m_3 \). According to Proposition 2.2 it turns out that \( G \) belongs to the class (20) or (31) if \( m_3 = 5m_2 \). \( \square \)

Proposition 2.3. Let \( G \) be a connected strongly regular graph of order \( n \) and degree \( r \) with \( m_2 = 6m_3 \). Then \( G \) belongs to the class (40) or (50) or (60) or (70) or (80) or (90) represented in Theorem 2.2.

Proof. Let \( m_3 = p \), \( m_2 = 6p \) and \( n = 7p + 1 \), where \( p \in \mathbb{N} \). Using (1.2) we obtain \( r = p(|\lambda_3| - 6\lambda_2) \). Let \( |\lambda_3| - 6\lambda_2 = t \), where \( t = 1, 2, \ldots, 6 \). Let \( \lambda_2 = k \), where \( k \) is a positive integer. Then according to Remark 1.3 we have (i) \( \lambda_3 = -(6k + t) \); (ii) \( \tau - \theta = -(5k + t) \); (iii) \( \delta = 7k + t \); (iv) \( r = pt \) and (v) \( \theta = pt - 6k^2 - kt \). In this case we can easily see that (1.3) is reduced to

\[
(p + 1)^2 - (7p + 1)t + 42k^2 + 12kt = 0. \tag{2.3}
\]

Case 1. \( (t = 1) \). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = k \) and \( \lambda_3 = -(6k + 1) \), \( \tau - \theta = -(5k + 1) \), \( \delta = 7k + 1 \), \( r = p \) and \( \theta = p - 6k^2 - k \). Using (2.3) we find that \( p = k(7k + 2) \). So we obtain that \( G \) is a strongly regular graph of order \( n = (7k + 1)^2 \) and degree \( r = k(7k + 2) \) with \( \tau = k^2 - 4k - 1 \) and \( \theta = k(k+1) \).

Case 2. \( (t = 2) \). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = k \) and \( \lambda_3 = -(6k + 2) \), \( \tau - \theta = -(5k + 2) \), \( \delta = 7k + 2 \), \( r = 2p \) and \( \theta = 2p - 6k^2 - 2k \). Using (2.3) we find that \( 5p - 1 = 3k(7k + 4) \). Replacing \( k \) with \( 5k - 1 \) we arrive at \( p = 105k^2 - 30k + 2 \). So we obtain that \( G \) is a strongly regular graph of order \( n = 15(7k - 1)^2 \) and degree \( r = 2(105k^2 - 30k + 2) \) with \( \tau = 60k^2 - 35k + 3 \) and \( \theta = 10k(6k - 1) \).

Case 3. \( (t = 3) \). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = k \) and \( \lambda_3 = -(6k + 3) \), \( \tau - \theta = -(5k + 3) \), \( \delta = 7k + 3 \), \( r = 3p \) and \( \theta = 3p - 6k^2 - 3k \). Using
(2.3) we find that $2p - 1 = k(7k + 6)$. Replacing $k$ with $2k - 1$ we arrive at $p = 14k^2 - 8k + 1$. So we obtain that $G$ is a strongly regular graph of order $n = 2(7k - 2)^2$ and degree $r = 3(14k^2 - 8k + 1)$ with $\tau = 18k^2 - 16k + 2$ and $\theta = 6k(3k - 1)$.

**Case 4.** ($t = 4$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(6k + 4)$, $\tau - \theta = -(5k + 4)$, $\delta = 7k + 4$, $r = 4p$ and $\theta = 4p - 6k^2 - 4k$. Using (2.3) we find that $2p - 2 = k(7k + 8)$. Replacing $k$ with $2k$ we arrive at $p = 14k^2 + 8k + 1$. So we obtain that $G$ is a strongly regular graph of order $n = 2(7k + 2)^2$ and degree $r = 4(14k^2 + 8k + 1)$ with $\tau = 2k(16k + 7)$ and $\theta = 4(2k + 1)(4k + 1)$.

**Case 5.** ($t = 5$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(6k + 5)$, $\tau - \theta = -(5k + 5)$, $\delta = 7k + 5$, $r = 6p$ and $\theta = 6p - 6k^2 - 6k$. Using (2.3) we find that $5p - 10 = 3k(7k + 10)$. Replacing $k$ with $5k$ we arrive at $p = 105k^2 + 30k + 2$. So we obtain that $G$ is a strongly regular graph of order $n = 15(7k + 1)^2$ and degree $r = 5(105k^2 + 30k + 2)$ with $\tau = 5(5k + 1)(15k + 1)$ and $\theta = 5(5k + 1)(15 + 2)$.

**Case 6.** ($t = 6$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(6k + 6)$, $\tau - \theta = -(5k + 6)$, $\delta = 7k + 6$, $r = 6p$ and $\theta = 6p - 6k^2 - 6k$. Using (2.3) we find that $p = (k + 1)(7k + 5)$. Replacing $k$ with $k - 1$ we arrive at $p = k(7k - 2)$. So we obtain that $G$ is a strongly regular graph of order $n = (7k - 1)^2$ and degree $r = 6k(7k - 2)$ with $\tau = 36k^2 - 11k - 1$ and $\theta = 6k(6k - 1)$.

**Proposition 2.4.** Let $G$ be a connected strongly regular graph of order $n$ and degree $r$ with $m_3 = 6m_2$. Then $G$ belongs to the class $(4^0)$ or $(5^0)$ or $(6^0)$ or $(7^0)$ or $(8^0)$ or $(9^0)$ represented in Theorem 2.2.

**Proof.** Let $m_2 = p$, $m_3 = 6p$ and $n = 7p + 1$, where $p \in \mathbb{N}$. Using (1.2) we obtain $r = p(6|\lambda_3| - \lambda_2)$. Let $6|\lambda_3| - \lambda_2 = t$, where $t = 1, 2, \ldots, 6$. Let $\lambda_3 = -k$, where $k$ is a positive integer. Then according to Remark 1.3 we have (i) $\lambda_2 = 6k - t$; (ii) $\tau - \theta = 5k - t$; (iii) $\delta = 7k - t$; (iv) $r = pt$ and (v) $\theta = pt - 6k^2 + kt$. In this case we can easily see that (1.4) is reduced to

$$
(p + 1)^2 - (7p + 1)t + 42k^2 - 12kt = 0. \tag{2.4}
$$

**Case 1.** ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 6k - 1$ and $\lambda_3 = -k$, $\tau - \theta = 5k - 1$, $\delta = 7k - 1$, $r = p$ and $\theta = p - 6k^2 + k$. Using (2.4) we find that $p = k(7k - 2)$. So we obtain that $G$ is a strongly regular graph of order $n = (7k - 1)^2$ and degree $r = k(7k - 2)$ with $\tau = k^2 + 4k - 1$ and $\theta = k(k - 1)$.

**Case 2.** ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 6k - 2$ and $\lambda_3 = -k$, $\tau - \theta = 5k - 2$, $\delta = 7k - 2$, $r = 2p$ and $\theta = 2p - 6k^2 + 2k$. Using (2.4) we find that $5p - 1 = 3k(7k - 4)$. Replacing $k$ with $5k + 1$ we arrive at $p = 105k^2 + 30k + 2$. So we obtain that $G$ is a strongly regular graph of order $n = 15(7k + 1)^2$ and degree $r = 2(105k^2 + 30k + 2)$ with $\tau = 60k^2 + 35k + 3$ and $\theta = 10k(6k + 1)$.

**Case 3.** ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 6k - 3$ and $\lambda_3 = -k$, $\tau - \theta = 5k - 3$, $\delta = 7k - 3$, $r = 3p$ and $\theta = 3p - 6k^2 + 3k$. Using (2.4) we
find that $2p - 1 = k(7k - 6)$. Replacing $k$ with $2k + 1$ we arrive at $p = 14k^2 + 8k + 1$. So we obtain that $G$ is a strongly regular graph of order $n = 2(7k + 2)^2$ and degree $r = 3(14k^2 + 8k + 1)$ with $\tau = 18k^2 + 16k + 2$ and $\theta = 6k(3k + 1)$.

**Case 4.** ($t = 4$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 6k - 4$ and $\lambda_3 = -k$, $\tau - \theta = 5k - 4$, $\delta = 7k - 4$, $r = 4p$ and $\theta = 4p - 6k^2 + 4k$. Using (2.4) we find that $2p - 2 = k(7k - 8)$. Replacing $k$ with $2k$ we arrive at $p = 14k^2 - 8k + 1$. So we obtain that $G$ is a strongly regular graph of order $n = 2(7k - 2)^2$ and degree $r = 4(14k^2 - 8k + 1)$ with $\tau = 2k(16k - 7)$ and $\theta = 4(2k - 1)(4k - 1)$.

**Case 5.** ($t = 5$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 6k - 5$ and $\lambda_3 = -k$, $\tau - \theta = 5k - 5$, $\delta = 7k - 5$, $r = 5p$ and $\theta = 5p - 6k^2 + 5k$. Using (2.4) we find that $5p - 10 = 3k(7k - 10)$. Replacing $k$ with $5k$ we arrive at $p = 105k^2 - 30k + 2$. So we obtain that $G$ is a strongly regular graph of order $n = 15(7k - 1)^2$ and degree $r = 5(105k^2 - 30k + 2)$ with $\tau = 5(5k - 1)(15k - 1)$ and $\theta = 5(5k - 1)(15k - 2)$.

**Case 6.** ($t = 6$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 6k - 6$ and $\lambda_3 = -k$, $\tau - \theta = 5k - 6$, $\delta = 7k - 6$, $r = 6p$ and $\theta = 6p - 6k^2 + 6k$. Using (2.4) we find that $p = (k - 1)(7k - 5)$. Replacing $k$ with $k + 1$ we arrive at $p = k(7k + 2)$. So we obtain that $G$ is a strongly regular graph of order $n = (7k + 1)^2$ and degree $r = 6k(7k + 2)$ with $\tau = 36k^2 + 11k - 1$ and $\theta = 6k(6k + 1)$.

**Remark 2.6.** We note that the complete bipartite graph $K_{4,4}$ is a strongly regular graph with $m_2 = 6m_3$. It is obtained from the class Theorem 2.2 ($\pi_0$) for $k = 0$.

**Remark 2.7.** We note that $3K_5$ is a strongly regular graph with $m_2 = 6m_3$. It is obtained from the class Theorem 2.2 ($\pi_0$) for $k = 0$.

**Remark 2.8.** We note that $6K_6$ is a strongly regular graph with $m_2 = 6m_3$. It is obtained from the class Theorem 2.2 ($\pi_0$) for $k = 1$.

**Theorem 2.2.** Let $G$ be a connected strongly regular graph of order $n$ and degree $r$ with $m_2 = 6m_3$ or $m_3 = 6m_2$. Then $G$ is one of the following strongly regular graphs:

1. $G$ is the complete bipartite graph $K_{4,4}$ of order $n = 8$ and degree $r = 4$ with $\tau = 0$ and $\theta = 4$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -4$ with $m_2 = 6$ and $m_3 = 1$.
2. $G$ is the strongly regular graph $\overline{3K_5}$ of order $n = 15$ and degree $r = 10$ with $\tau = 5$ and $\theta = 10$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -5$ with $m_2 = 12$ and $m_3 = 2$.
3. $G$ is the strongly regular graph $\overline{6K_6}$ of order $n = 36$ and degree $r = 30$ with $\tau = 24$ and $\theta = 30$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -6$ with $m_2 = 30$ and $m_3 = 5$.
4. $G$ is a strongly regular graph of order $n = (7k - 1)^2$ and degree $r = k(7k - 2)$ with $\tau = k^2 + 4k - 1$ and $\theta = k(k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = 6k - 1$ and $\lambda_3 = -k$ with $m_2 = k(7k - 2)$ and $m_3 = 6k(7k - 2)$. 

(4°) G is a strongly regular graph of order $n = (7k-1)^2$ and degree $r = 6k(7k-2)$ with $\tau = 36k^2-11k-1$ and $\theta = 6k(6k-1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = k-1$ and $\lambda_3 = -6k$ with $m_2 = 6k(7k-2)$ and $m_3 = k(7k-2)$; 

(5°) G is a strongly regular graph of order $n = (7k+1)^2$ and degree $r = k(7k+2)$ with $\tau = k^2-4k-1$ and $\theta = k(k+1)$, where $k \geq 5$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(6k+1)$ with $m_2 = 6k(7k+2)$ and $m_3 = k(7k+2)$; 

(6°) G is a strongly regular graph of order $n = (7k+1)^2$ and degree $r = 6k(7k+2)$ with $\tau = 36k^2+11k-1$ and $\theta = 6k(6k+1)$, where $k \geq 5$. Its eigenvalues are $\lambda_2 = 6k$ and $\lambda_3 = -(k+1)$ with $m_2 = k(7k+2)$ and $m_3 = 6k(7k+2)$; 

(7°) G is a strongly regular graph of order $n = 2(7k-2)^2$ and degree $r = 3(14k^2-8k+1)$ with $\tau = 18k^2-16k+2$ and $\theta = 6k(3k-1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 12k-4$ and $\lambda_3 = -2k$ with $m_2 = 14k^2-8k+1$ and $m_3 = 6(14k^2-8k+1)$; 

(8°) G is a strongly regular graph of order $n = 2(7k+2)^2$ and degree $r = 3(14k^2+8k+1)$ with $\tau = 18k^2+16k+2$ and $\theta = 6k(3k+1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 12k+3$ and $\lambda_3 = -(2k+1)$ with $m_2 = 14k^2+8k+1$ and $m_3 = 6(14k^2+8k+1)$; 

(9°) G is a strongly regular graph of order $n = 15(7k-1)^2$ and degree $r = 2(105k^2-30k+2)$ with $\tau = 60k^2-35k+3$ and $\theta = 10k(6k-1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 5k-1$ and $\lambda_3 = -(30k-4)$ with $m_2 = 6(105k^2-30k+2)$ and $m_3 = 105k^2-30k+2$; 

(10°) G is a strongly regular graph of order $n = 15(7k-1)^2$ and degree $r = 5(105k^2-30k+2)$ with $\tau = 5(5k-1)(15k-1)$ and $\theta = 5(5k-1)(15k-2)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 30k-5$ and $\lambda_3 = -5k$ with $m_2 = 105k^2-30k+2$ and $m_3 = 6(105k^2-30k+2)$; 

(11°) G is a strongly regular graph of order $n = 15(7k+1)^2$ and degree $r = 2(105k^2+30k+2)$ with $\tau = 60k^2+35k+3$ and $\theta = 10k(6k+1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 30k+4$ and $\lambda_3 = -(5k+1)$ with $m_2 = 105k^2+30k+2$ and $m_3 = 6(105k^2+30k+2)$; 

(12°) G is a strongly regular graph of order $n = 15(7k+1)^2$ and degree $r = 5(105k^2+30k+2)$ with $\tau = 5(5k+1)(15k+1)$ and $\theta = 5(5k+1)(15+2)$,
where \( k \in \mathbb{N} \). Its eigenvalues are \( \lambda_2 = 5k \) and \( \lambda_3 = -(30k+5) \) with \( m_2 = 6(105k^2 + 30k+2) \) and \( m_3 = 105k^2 + 30k+2 \).

**Proof.** Firstly, according to Remark 2.4 we have \( \alpha(\beta - 1) = 6(\alpha - 1) \), from which we find that \( \alpha = 2 \), \( \beta = 4 \) or \( \alpha = 3 \), \( \beta = 5 \) or \( \alpha = 6 \), \( \beta = 6 \). In view of this we obtain the strongly regular graphs represented in Theorem 2.2 \((1^0)\), \((2^0)\), \((3^0)\).

Next, according to Proposition 2.3 it turns out that \( G \) belongs to the class \((4^0)\) or \((5^0)\) or \((6^0)\) or \((7^0)\) or \((8^0)\) or \((9^0)\) if \( m_2 = 6m_3 \). According to Proposition 2.4 it turns out that \( G \) belongs to the class \((4^0)\) or \((5^0)\) or \((6^0)\) or \((7^0)\) or \((8^0)\) or \((9^0)\) if \( m_3 = 6m_2 \).

**Proposition 2.5.** Let \( G \) be a connected strongly regular graph of order \( n \) and degree \( r \) with \( m_2 = 7m_3 \). Then \( G \) belongs to the class \((2^0)\) or \((3^0)\) or \((4^0)\) or \((5^0)\) represented in Theorem 2.3.

**Proof.** Let \( m_3 = p \), \( m_2 = 7p \) and \( n = 8p + 1 \), where \( p \in \mathbb{N} \). Using (1.2) we obtain\( r = p(\mid\lambda_3\mid - 7\lambda_2) \). Let \( \mid\lambda_3\mid - 7\lambda_2 = t \), where \( t = 1, 2, \ldots, 7 \). Let \( \lambda_2 = k \), where \( k \) is a positive integer. Then according to Remark 1.3 we have (i) \( \lambda_2 = -(7k+t) \); (ii) \( \tau - \theta = -(6k+t) \); (iii) \( \delta = 8k+t \); (iv) \( r = pt \) and (v) \( \theta = pt - 7k^2 - kt \). In this case we can easily see that (1.3) is reduced to

\[(p+1)t^2 - (8p+1)t + 56k + 14kt = 0. \tag{2.5}\]

**Case 1.** \((t = 1)\). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = k \) and \( \lambda_3 = -(7k+1) \), \( \tau - \theta = -(6k+1) \), \( \delta = 8k+1 \), \( r = p \) and \( \theta = p - 7k^2 - k \). Using (2.5) we find that \( p = 2k(4k+1) \). So we obtain that \( G \) is a strongly regular graph of order \( n = (8k+1)^2 \) and degree \( r = 2k(4k+1) \) with \( \tau = k^2 - 5k - 1 \) and \( \theta = k(k+1) \).

**Case 2.** \((t = 2)\). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = k \) and \( \lambda_3 = -(7k+2) \), \( \tau - \theta = -(6k+2) \), \( \delta = 8k+2 \), \( r = 2p \) and \( \theta = 2p - 7k^2 - 2k \). Using (2.5) we find that 6p - 1 = 14k(2k+1), a contradiction because 2 \mid 6p - 1.

**Case 3.** \((t = 3)\). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = k \) and \( \lambda_3 = -(7k+3) \), \( \tau - \theta = -(6k+3) \), \( \delta = 8k+3 \), \( r = 3p \) and \( \theta = 3p - 7k^2 - 3k \). Using (2.5) we find that 15p - 6 = 14k(4k+3). Replacing \( k \) with 15k - 6 we arrive at \( p = 840k^2 - 630k + 118 \). So we obtain that \( G \) is a strongly regular graph of order \( n = 105(8k-3)^2 \) and degree \( r = 3(840k^2 - 630k + 118) \) with \( \tau = 945k^2 - 765k + 153 \) and \( \theta = 15(3k-1)(21k-8) \).

**Case 4.** \((t = 4)\). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = k \) and \( \lambda_3 = -(7k+4) \), \( \tau - \theta = -(6k+4) \), \( \delta = 8k+4 \), \( r = 4p \) and \( \theta = 4p - 7k^2 - 4k \). Using (2.5) we find that 4p - 3 = 14k(k+1), a contradiction because 2 \mid 4p - 3.

**Case 5.** \((t = 5)\). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = k \) and \( \lambda_3 = -(7k+5) \), \( \tau - \theta = -(6k+5) \), \( \delta = 8k+5 \), \( r = 5p \) and \( \theta = 5p - 7k^2 - 5k \). Using (2.5) we find that 15p - 20 = 14k(4k+5). Replacing \( k \) with 15k + 5 we arrive at \( p = 840k^2 + 630k + 118 \). So we obtain that \( G \) is a strongly regular graph of order
$n = 105(8k+3)^2$ and degree $r = 5(840k^2 + 630k + 118)$ with $\tau = 5(525k^2 + 387k + 71)$ and $\theta = 15(5k+2)(35k+13)$.

**Case 6.** ($t = 6$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(7k+6)$, $\tau - \theta = -(6k+6)$, $\delta = 8k+6$, $r = 6p$ and $\theta = 6p - 7k^2 - 6k$. Using (2.5) we find that $6p - 15 = 14k(2k+3)$, a contradiction because $2 \nmid 6p - 15$.

**Case 7.** ($t = 7$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(7k+7)$, $\tau - \theta = -(6k+7)$, $\delta = 8k+7$, $r = 7p$ and $\theta = 7p - 7k^2 - 7k$. Using (2.5) we find that $p = 2(k+1)(4k + 3)$. Replacing $k$ with $k - 1$ we arrive at $p = 2k(4k - 1)$. So we obtain that $G$ is a strongly regular graph of order $n = (8k-1)^2$ and degree $r = 14k(4k-1)$ with $\tau = 49k^2 - 13k - 1$ and $\theta = 7k(7k - 1)$.

**Proposition 2.6.** Let $G$ be a connected strongly regular graph of order $n$ and degree $r$ with $m_3 = 7m_2$. Then $G$ belongs to the class $(2^0)$ or $(3^0)$ or $(4^0)$ or $(5^0)$ represented in Theorem 2.3.

**Proof.** Let $m_2 = p$, $m_3 = 7p$ and $n = 8p + 1$, where $p \in \mathbb{N}$. Using (1.2) we obtain $r = p(7|\lambda_3| - \lambda_2)$. Let $7|\lambda_3| - \lambda_2 = t$, where $t = 1, 2, \ldots, 7$. Let $\lambda_3 = -k$, where $k$ is a positive integer. Then according to Remark 1.3 we have (i) $\lambda_2 = 7k - t$; (ii) $\tau - \theta = 6k - t$; (iii) $\delta = 8k - t$; (iv) $r = pt$ and (v) $\theta = pt - 7k^2 + kt$. In this case we can easily see that (1.4) is reduced to

$$(p+1)t^2 - (8p+1)t + 56k^2 - 14kt = 0.$$  

**Case 1.** ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 7k - 1$ and $\lambda_3 = -k$, $\tau - \theta = 6k - 1$, $\delta = 8k - 1$, $r = p$ and $\theta = 7k^2 + k$. Using (2.6) we find that $p = 2k(4k - 1)$. So we obtain that $G$ is a strongly regular graph of order $n = (8k-1)^2$ and degree $r = 2k(4k - 1)$ with $\tau = k^2 + 5k - 1$ and $\theta = k(k - 1)$.

**Case 2.** ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 7k - 2$ and $\lambda_3 = -k$, $\tau - \theta = 6k - 2$, $\delta = 8k - 2$, $r = 2p$ and $\theta = 2p - 7k^2 + 2k$. Using (2.6) we find that $6p - 1 = 14k(2k - 1)$, a contradiction because $2 \nmid 6p - 1$.

**Case 3.** ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 7k - 3$ and $\lambda_3 = -k$, $\tau - \theta = 6k - 3$, $\delta = 8k - 3$, $r = 3p$ and $\theta = 3p - 7k^2 + 3k$. Using (2.6) we find that $15p - 6 = 14k(4k - 3)$. Replacing $k$ with $15k + 6$ we arrive at $p = 840k^2 + 630k + 118$. So we obtain that $G$ is a strongly regular graph of order $n = 105(8k+3)^2$ and degree $r = 3(840k^2 + 630k + 118)$ with $\tau = 945k^2 + 765k + 153$ and $\theta = 15(3k+1)(21k+8)$.

**Case 4.** ($t = 4$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 7k - 4$ and $\lambda_3 = -k$, $\tau - \theta = 6k - 4$, $\delta = 8k - 4$, $r = 4p$ and $\theta = 4p - 7k^2 + 4k$. Using (2.6) we find that $4p - 3 = 14k(k - 1)$, a contradiction because $2 \nmid 4p - 3$.

**Case 5.** ($t = 5$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 7k - 5$ and $\lambda_3 = -k$, $\tau - \theta = 6k - 5$, $\delta = 8k - 5$, $r = 5p$ and $\theta = 5p - 7k^2 + 5k$. Using (2.6) we find that $15p - 20 = 14k(4k - 5)$. Replacing $k$ with $15k - 5$ we arrive at $p = 840k^2 - 630k + 118$. So we obtain that $G$ is a strongly regular graph of order $n =$
105(8k - 3)^2 and degree r = 5(840k^2 - 630k + 118) with \( \tau = 5(525k^2 - 387k + 71) \) and \( \theta = 15(5k - 2)(35k - 13) \).

**Case 6.** \((t = 6)\). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = 7k - 6 \) and \( \lambda_3 = -k \), \( \tau - \theta = 6k - 6 \), \( \delta = 8k - 6 \), \( r = 6p \) and \( \theta = 6p - 7k^2 + 6k \). Using (2.6) we find that \( 6p - 15 = 14k(2k - 3) \), a contradiction because \( 2 | 6p - 15 \).

**Case 7.** \((t = 7)\). Using (i), (ii), (iii), (iv) and (v) we find that \( \lambda_2 = 7k - 7 \) and \( \lambda_3 = -k \), \( \tau - \theta = 6k - 7 \), \( \delta = 8k - 7 \), \( r = 7p \) and \( \theta = 7p - 7k^2 + 7k \). Using (2.6) we find that \( p = 2(k - 1)(4k - 3) \). Replacing \( k \) with \( k + 1 \) we arrive at \( p = 2k(4k + 1) \). So we obtain that \( G \) is a strongly regular graph of order \( n = (8k + 1)^2 \) and degree \( r = 14k(4k + 1) \) with \( \tau = 49k^2 + 13k - 1 \) and \( \theta = 7k(7k + 1) \). \( \square \)

**Remark 2.9.** We note that \( \overline{7K_7} \) is a strongly regular graph with \( m_2 = 7m_3 \). It is obtained from the class Theorem 2.3 (2^0) for \( k = 1 \).

**Theorem 2.3.** Let \( G \) be a connected strongly regular graph of order \( n \) and degree \( r \) with \( m_2 = 7m_3 \) or \( m_3 = 7m_2 \). Then \( G \) is one of the following strongly regular graphs:

1. **(1)** \( G \) is the strongly regular graph \( \overline{7K_7} \) of order \( n = 49 \) and degree \( r = 42 \) with \( \tau = 35 \) and \( \theta = 42 \). Its eigenvalues are \( \lambda_2 = 0 \) and \( \lambda_3 = -7 \) with \( m_2 = 42 \) and \( m_3 = 6 \);

2. **(2)** \( G \) is a strongly regular graph of order \( n = (8k - 1)^2 \) and degree \( r = 2k(4k - 1) \) with \( \tau = k^2 + 5k - 1 \) and \( \theta = k(k - 1) \), where \( k \geq 2 \). Its eigenvalues are \( \lambda_2 = 7k - 1 \) and \( \lambda_3 = -k \) with \( m_2 = 2k(4k - 1) \) and \( m_3 = 14k(4k - 1) \);

3. **(3)** \( G \) is a strongly regular graph of order \( n = (8k - 1)^2 \) and degree \( r = 14k(4k - 1) \) with \( \tau = 49k^2 - 13k - 1 \) and \( \theta = 7k(7k - 1) \), where \( k \geq 2 \). Its eigenvalues are \( \lambda_2 = k - 1 \) and \( \lambda_3 = -7k \) with \( m_2 = 14k(4k - 1) \) and \( m_3 = 2k(4k - 1) \);

4. **(4)** \( G \) is a strongly regular graph of order \( n = (8k + 1)^2 \) and degree \( r = 2k(4k + 1) \) with \( \tau = k^2 + 5k - 1 \) and \( \theta = k(k + 1) \), where \( k \geq 6 \). Its eigenvalues are \( \lambda_2 = k \) and \( \lambda_3 = -(7k + 1) \) with \( m_2 = 14k(4k + 1) \) and \( m_3 = 2k(4k + 1) \);

5. **(5)** \( G \) is a strongly regular graph of order \( n = (8k + 1)^2 \) and degree \( r = 14k(4k + 1) \) with \( \tau = 49k^2 + 13k - 1 \) and \( \theta = 7k(7k + 1) \), where \( k \geq 6 \). Its eigenvalues are \( \lambda_2 = 7k \) and \( \lambda_3 = -(k + 1) \) with \( m_2 = 2k(4k + 1) \) and \( m_3 = 14k(4k + 1) \);

6. **(6)** \( G \) is a strongly regular graph of order \( n = 105(8k - 3)^2 \) and degree \( r = 3(840k^2 - 630k + 118) \) with \( \tau = 945k^2 - 765k + 153 \) and \( \theta = 15(3k - 1)(21k - 8) \), where \( k \in \mathbb{N} \). Its eigenvalues are \( \lambda_2 = 15k - 6 \) and \( \lambda_3 = -(105k - 39) \) with \( m_2 = 7(840k^2 - 630k + 118) \) and \( m_3 = 840k^2 - 630k + 118 \);

7. **(7)** \( G \) is a strongly regular graph of order \( n = 105(8k - 3)^2 \) and degree \( r = 5(840k^2 - 630k + 118) \) with \( \tau = 5(525k^2 - 387k + 71) \) and \( \theta = 15(5k - 2)(35k - 13) \), where \( k \in \mathbb{N} \). Its eigenvalues are \( \lambda_2 = 105k - 40 \) and \( \lambda_3 = -(15k - 5) \) with \( m_2 = 840k^2 - 630k + 118 \) and \( m_3 = 7(840k^2 - 630k + 118) \);

8. **(8)** \( G \) is a strongly regular graph of order \( n = 105(8k + 3)^2 \) and degree \( r = 3(840k^2 + 630k + 118) \) with \( \tau = 945k^2 + 765k + 153 \) and \( \theta = 15(3k + 1)(21k + 13) \).
8, where $k \geq 0$. Its eigenvalues are $\lambda_2 = 105k + 39$ and $\lambda_3 = -(15k + 6)$ with $m_2 = 840k^2 + 630k + 118$ and $m_3 = 7(840k^2 + 630k + 118)$.

(5.0) $G$ is a strongly regular graph of order $n = 105(8k + 3)^2$ and degree $r = 5(840k^2 + 630k + 118)$ with $\tau = 5(525k^2 + 387k + 71)$ and $\theta = 15(5k + 2)(35k + 13)$, where $k \geq 0$. Its eigenvalues are $\lambda_2 = 15k + 5$ and $\lambda_3 = -(105k + 40)$ with $m_2 = 7(840k^2 + 630k + 118)$ and $m_3 = 840k^2 + 630k + 118$.

Proof. Firstly, according to Remark 2.4 we have $\alpha(\beta - 1) = 7(\alpha - 1)$, from which we find that $\alpha = 7$, $\beta = 7$. In view of this we obtain the strongly regular graph represented in Theorem 2.3 (10). Next, according to Proposition 2.5 it turns out that $G$ belongs to the class $(2^0)$ or $(3^0)$ or $(4^0)$ or $(5^0)$ if $m_2 = 7m_3$. According to Proposition 2.6 it turns out that $G$ belongs to the class $(2^0)$ or $(3^0)$ or $(4^0)$ or $(5^0)$ if $m_3 = 7m_2$. \qed

Proposition 2.7. Let $G$ be a connected strongly regular graph of order $n$ and degree $r$ with $m_2 = 8m_3$. Then $G$ belongs to the class $(4^0)$ or $(5^0)$ or $(6^0)$ or $(7^0)$ or $(8^0)$ or $(9^0)$ or $(10^0)$ or $(11^0)$ represented in Theorem 2.4.

Proof. Let $m_3 = p$, $m_2 = 8p$ and $n = 9p + 1$, where $p \in \mathbb{N}$. Using (1.2) we obtain $r = p(|\lambda_3| - 8\lambda_2)$. Let $|\lambda_3| - 8\lambda_2 = t$, where $t = 1, 2, \ldots, 8$. Let $\lambda_2 = k$, where $k$ is a positive integer. Then according to Remark 1.3 we have (i) $\lambda_3 = -(8k + t)$; (ii) $\tau - \theta = -(7k + t)$; (iii) $\delta = 9k + t$; (iv) $r = pt$ and (v) $\theta = pt - 8k^2 - kt$. In this case we can easily see that (1.3) is reduced to

$$\frac{(p + 1)^2 - (9p + 1)t + 72k^2 + 16kt}{2} = 0. \quad (2.7)$$

Case 1. ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(8k + 1)$, $\tau - \theta = -(7k + 1)$, $\delta = 9k + 1$, $r = p$ and $\theta = p - 8k^2 - k$. Using (2.7) we find that $p = k(9k + 2)$. So we obtain that $G$ is a strongly regular graph of order $n = (9k + 1)^2$ and degree $r = k(9k + 2)$ with $\tau = k^2 - 6k - 1$ and $\theta = k(k + 1)$.

Case 2. ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(8k + 2)$, $\tau - \theta = -(7k + 2)$, $\delta = 9k + 2$, $r = 2p$ and $\theta = 2p - 8k^2 - 2k$. Using (2.7) we find that $7p - 1 = 4k(9k + 4)$. Replacing $k$ with $7k - 1$ we arrive at $p = 252k^2 - 56k + 3$. So we obtain that $G$ is a strongly regular graph of order $n = 28(9k - 1)^2$ and degree $r = 2(252k^2 - 56k + 3)$ with $\tau = 112k^2 - 63k + 5$ and $\theta = 14k(8k - 1)$.

Case 3. ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(8k + 3)$, $\tau - \theta = -(7k + 3)$, $\delta = 9k + 3$, $r = 3p$ and $\theta = 3p - 8k^2 - 3k$. Using (2.7) we find that $3p - 1 = 4k(3k + 2)$. Replacing $k$ with $3k + 1$ we arrive at $p = (2k + 1)(18k + 7)$. So we obtain that $G$ is a strongly regular graph of order $n = 4(9k + 4)^2$ and degree $r = 3(2k + 1)(18k + 7)$ with $\tau = 18k(2k + 1)$ and $\theta = (3k + 2)(12k + 5)$.

Case 4. ($t = 4$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(8k + 4)$, $\tau - \theta = -(7k + 4)$, $\delta = 9k + 4$, $r = 4p$ and $\theta = 4p - 8k^2 - 4k$. Using
(2.7) we find that $5p - 3 = 2k(9k + 8)$. Replacing $k$ with $5k - 1$ we arrive at $p = 90k^2 - 20k + 1$. So we obtain that $G$ is a strongly regular graph of order $n = 10(9k - 1)^2$ and degree $r = 4(90k^2 - 20k + 1)$ with $\tau = 160k^2 - 55k + 3$ and $\theta = 20k(8k - 1)$.

Case 5. ($t = 5$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(8k + 5)$, $\tau - \theta = -(7k + 5)$, $\delta = 9k + 5$, $r = 5p$ and $\theta = 5p - 8k^2 - 5k$. Using (2.7) we find that $5p - 5 = 2k(9k + 10)$. Replacing $k$ with $5k$ we arrive at $p = 90k^2 + 20k + 1$. So we obtain that $G$ is a strongly regular graph of order $n = 10(9k + 1)^2$ and degree $r = 5(90k^2 + 20k + 1)$ with $\tau = 10k(25k + 4)$ and $\theta = 5(5k + 1)(10k + 1)$.

Case 6. ($t = 6$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(8k + 6)$, $\tau - \theta = -(7k + 6)$, $\delta = 9k + 6$, $r = 6p$ and $\theta = 6p - 8k^2 - 6k$. Using (2.7) we find that $3p - 5 = 4k(3k + 4)$. Replacing $k$ with $3k - 2$ we arrive at $p = (2k - 1)(18k - 7)$. So we obtain that $G$ is a strongly regular graph of order $n = 4(9k - 4)^2$ and degree $r = 6(2k - 1)(18k - 7)$ with $\tau = 3(48k^2 - 45k + 10)$ and $\theta = 2(3k - 1)(24k - 11)$.

Case 7. ($t = 7$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(8k + 7)$, $\tau - \theta = -(7k + 7)$, $\delta = 9k + 7$, $r = 7p$ and $\theta = 7p - 8k^2 - 7k$. Using (2.7) we find that $7p - 21 = 4k(9k + 14)$. Replacing $k$ with $7k$ we arrive at $p = 252k^2 + 56k + 3$. So we obtain that $G$ is a strongly regular graph of order $n = 28(9k + 1)^2$ and degree $r = 7(252k^2 + 56k + 3)$ with $\tau = 14(7k + 1)(14k + 1)$ and $\theta = 7(7k + 1)(28k + 3)$.

Case 8. ($t = 8$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(8k + 8)$, $\tau - \theta = -(7k + 8)$, $\delta = 9k + 8$, $r = 8p$ and $\theta = 8p - 8k^2 - 8k$. Using (2.7) we find that $p = (k + 1)(9k + 7)$. Replacing $k$ with $k - 1$ we arrive at $p = k(9k - 2)$. So we obtain that $G$ is a strongly regular graph of order $n = (9k - 1)^2$ and degree $r = 8k(9k - 2)$ with $\tau = 64k^2 - 15k - 1$ and $\theta = 8k(8k - 1)$.

□

Proposition 2.8. Let $G$ be a connected strongly regular graph of order $n$ and degree $r$ with $m_3 = 8m_2$. Then $G$ belongs to the class $(4^0)$ or $(5^0)$ or $(6^0)$ or $(7^0)$ or $(8^0)$ or $(9^0)$ or $(10^0)$ or $(11^0)$ represented in Theorem 2.4.

Proof. Let $m_2 = p$, $m_3 = 8p$ and $n = 9p + 1$, where $p \in \mathbb{N}$. Using (1.2) we obtain $r = p(8|\lambda_3| - \lambda_2)$. Let $8|\lambda_3| - \lambda_2 = t$, where $t = 1, 2, \ldots, 8$. Let $\lambda_3 = -k$, where $k$ is a positive integer. Then according to Remark 1.3 we have (i) $\lambda_2 = 8k - t$; (ii) $\tau - \theta = 7k - t$; (iii) $\delta = 9k - t$; (iv) $r = pt$ and (v) $\theta = pt - 8k^2 + kt$. In this case we can easily see that (1.4) is reduced to

$$
(p + 1)t^2 - (9p + 1)t + 72k^2 - 16kt = 0.
$$

(2.8)

Case 1. ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 8k - 1$ and $\lambda_3 = -k$, $\tau - \theta = 7k - 1$, $\delta = 9k - 1$, $r = p$ and $\theta = p - 8k^2 + k$. Using (2.8) we find that $p = k(9k - 2)$. So we obtain that $G$ is a strongly regular graph of order $n = (9k - 1)^2$ and degree $r = k(9k - 2)$ with $\tau = k^2 + 6k - 1$ and $\theta = k(k - 1)$.
Case 2. \((t = 2)\). Using (i), (ii), (iii), (iv) and (v) we find that \(\lambda_2 = 8k - 2\) and \(\lambda_3 = -k\), \(\tau - \theta = 7k - 2\), \(\delta = 9k - 2\), \(r = 2p\) and \(\theta = 2p - 8k^2 + 2k\). Using (2.8) we find that \(7p - 1 = 4k(9k - 4)\). Replacing \(k\) with \(7k + 1\) we arrive at \(p = 252k^2 + 56k + 3\). So we obtain that \(G\) is a strongly regular graph of order \(n = 28(9k + 1)^2\) and degree \(r = 2(252k^2 + 56k + 3)\) with \(\tau = 112k^2 + 63k + 5\) and \(\theta = 14k(8k + 1)\).

Case 3. \((t = 3)\). Using (i), (ii), (iii), (iv) and (v) we find that \(\lambda_2 = 8k - 3\) and \(\lambda_3 = -k\), \(\tau - \theta = 7k - 3\), \(\delta = 9k - 3\), \(r = 3p\) and \(\theta = 3p - 8k^2 + 3k\). Using (2.8) we find that \(3p - 1 = 4k(3k - 2)\). Replacing \(k\) with \(3k - 1\) we arrive at \(p = (2k - 1)(18k - 7)\). So we obtain that \(G\) is a strongly regular graph of order \(n = 4(9k - 4)^2\) and degree \(r = 3(2k - 1)(18k - 7)\) with \(\tau = 18k(2k - 1)\) and \(\theta = (3k - 2)(12k - 5)\).

Case 4. \((t = 4)\). Using (i), (ii), (iii), (iv) and (v) we find that \(\lambda_2 = 8k - 4\) and \(\lambda_3 = -k\), \(\tau - \theta = 7k - 4\), \(\delta = 9k - 4\), \(r = 4p\) and \(\theta = 4p - 8k^2 + 4k\). Using (2.8) we find that \(5p - 3 = 2k(9k - 8)\). Replacing \(k\) with \(5k + 1\) we arrive at \(p = 90k^2 + 20k + 1\). So we obtain that \(G\) is a strongly regular graph of order \(n = 10(9k + 1)^2\) and degree \(r = 4(90k^2 + 20k + 1)\) with \(\tau = 160k^2 + 55k + 3\) and \(\theta = 20k(8k + 1)\).

Case 5. \((t = 5)\). Using (i), (ii), (iii), (iv) and (v) we find that \(\lambda_2 = 8k - 5\) and \(\lambda_3 = -k\), \(\tau - \theta = 7k - 5\), \(\delta = 9k - 5\), \(r = 5p\) and \(\theta = 5p - 8k^2 + 5k\). Using (2.8) we find that \(5p - 5 = 2k(9k - 10)\). Replacing \(k\) with \(5k\) we arrive at \(p = 90k^2 - 20k + 1\). So we obtain that \(G\) is a strongly regular graph of order \(n = 10(9k - 1)^2\) and degree \(r = 5(90k^2 - 20k + 1)\) with \(\tau = 10k(25k - 4)\) and \(\theta = 5(5k - 1)(10k - 1)\).

Case 6. \((t = 6)\). Using (i), (ii), (iii), (iv) and (v) we find that \(\lambda_2 = 8k - 6\) and \(\lambda_3 = -k\), \(\tau - \theta = 7k - 6\), \(\delta = 9k - 6\), \(r = 6p\) and \(\theta = 6p - 8k^2 + 6k\). Using (2.8) we find that \(3p - 5 = 4k(3k - 4)\). Replacing \(k\) with \(3k + 2\) we arrive at \(p = (2k + 1)(18k + 7)\). So we obtain that \(G\) is a strongly regular graph of order \(n = 4(9k + 4)^2\) and degree \(r = 6(2k + 1)(18k + 7)\) with \(\tau = 3(48k^2 + 45k + 10)\) and \(\theta = 2(3k + 1)(24k + 11)\).

Case 7. \((t = 7)\). Using (i), (ii), (iii), (iv) and (v) we find that \(\lambda_2 = 8k - 7\) and \(\lambda_3 = -k\), \(\tau - \theta = 7k - 7\), \(\delta = 9k - 7\), \(r = 7p\) and \(\theta = 7p - 8k^2 + 7k\). Using (2.8) we find that \(7p - 21 = 4k(9k - 14)\). Replacing \(k\) with \(7k\) we arrive at \(p = 252k^2 - 56k + 3\). So we obtain that \(G\) is a strongly regular graph of order \(n = 28(9k - 1)^2\) and degree \(r = 7(252k^2 - 56k + 3)\) with \(\tau = 14(7k - 1)(14k - 1)\) and \(\theta = 7(7k - 1)(28k - 3)\).

Case 8. \((t = 8)\). Using (i), (ii), (iii), (iv) and (v) we find that \(\lambda_2 = 8k - 8\) and \(\lambda_3 = -k\), \(\tau - \theta = 7k - 8\), \(\delta = 9k - 8\), \(r = 8p\) and \(\theta = 8p - 8k^2 + 8k\). Using (2.8) we find that \(p = (k - 1)(9k - 7)\). Replacing \(k\) with \(k + 1\) we arrive at \(p = k(9k + 2)\). So we obtain that \(G\) is a strongly regular graph of order \(n = (9k + 1)^2\) and degree \(r = 8k(9k + 2)\) with \(\tau = 64k^2 + 15k - 1\) and \(\theta = 8k(8k + 1)\).

Remark 2.10. We note that the complete bipartite graph \(K_{5,5}\) is a strongly regular graph with \(m_2 = 8m_3\). It is obtained from the class Theorem 2.4 \((\overline{G}^0)\) for \(k = 0\).

Remark 2.11. We note that \(\overline{4K_2}\) is a strongly regular graph with \(m_2 = 8m_3\). It is obtained from the class Theorem 2.4 \((\overline{T}^0)\) for \(k = 0\).
Remark 2.12. We note that $8K_8$ is a strongly regular graph with $m_2 = 8m_3$. It is obtained from the class Theorem 2.4 ($\mathfrak{T}^0$) for $k = 1$.

**Theorem 2.4.** Let $G$ be a connected strongly regular graph of order $n$ and degree $r$ with $m_2 = 8m_3$ or $m_3 = 8m_2$. Then $G$ is one of the following strongly regular graphs:

1. $G$ is the complete bipartite graph $K_{5,5}$ of order $n = 10$ and degree $r = 5$ with $\tau = 0$ and $\theta = 5$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -5$ with $m_2 = 8$ and $m_3 = 1$;
2. $G$ is the strongly regular graph $4K_7$ of order $n = 28$ and degree $r = 21$ with $\tau = 14$ and $\theta = 21$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -7$ with $m_2 = 24$ and $m_3 = 3$;
3. $G$ is the strongly regular graph $8K_8$ of order $n = 64$ and degree $r = 56$ with $\tau = 48$ and $\theta = 56$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -8$ with $m_2 = 56$ and $m_3 = 7$;
4. $G$ is a strongly regular graph of order $n = (9k-1)^2$ and degree $r = k(9k-2)$ with $\tau = k^2+6k-1$ and $\theta = k(k-1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = 8k-1$ and $\lambda_3 = -k$ with $m_2 = k(9k-2)$ and $m_3 = 8k(9k-2)$;
5. $G$ is a strongly regular graph of order $n = (9k+1)^2$ and degree $r = k(9k+2)$ with $\tau = k^2-6k-1$ and $\theta = k(k+1)$, where $k \geq 7$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(8k+1)$ with $m_2 = 8k(9k+2)$ and $m_3 = k(9k+2)$;
6. $G$ is a strongly regular graph of order $n = (9k+2)^2$ and degree $r = 8k(9k+1)$ with $\tau = 64k^2+15k-1$ and $\theta = 8k(8k+1)$, where $k \geq 7$. Its eigenvalues are $\lambda_2 = 8k$ and $\lambda_3 = -(k+1)$ with $m_2 = k(9k+2)$ and $m_3 = 8k(9k+2)$;
7. $G$ is a strongly regular graph of order $n = 4(9k-4)^2$ and degree $r = 3(2k-1)(18k-7)$ with $\tau = 18k(2k-1)$ and $\theta = (3k-2)(12k-5)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 24k-11$ and $\lambda_3 = -(3k-1)$ with $m_2 = (2k-1)(18k-7)$ and $m_3 = 8(2k-1)(18k-7)$;
8. $G$ is a strongly regular graph of order $n = 4(9k-4)^2$ and degree $r = 6(2k-1)(18k-7)$ with $\tau = 3(48k^2-45k+10)$ and $\theta = 2(3k-1)(24k-11)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 3k-2$ and $\lambda_3 = -(24k-10)$ with $m_2 = 8(2k-1)(18k-7)$ and $m_3 = (2k-1)(18k-7)$;
9. $G$ is a strongly regular graph of order $n = 4(9k+4)^2$ and degree $r = 3(2k+1)(18k+7)$ with $\tau = 18k(2k+1)$ and $\theta = (3k+2)(12k+5)$, where $k \geq 0$. Its eigenvalues are $\lambda_2 = 3k+1$ and $\lambda_3 = -(24k+11)$ with $m_2 = 8(2k+1)(18k+7)$ and $m_3 = (2k+1)(18k+7)$;
10. $G$ is a strongly regular graph of order $n = 4(9k+4)^2$ and degree $r = 6(2k+1)(18k+7)$ with $\tau = 3(48k^2+45k+10)$ and $\theta = 2(3k+1)(24k+11)$, where
Firstly, according to Remark 2.4 we have that $G$ is a strongly regular graph of order $n = 10(9k-1)^2$ and degree $r = 4(90k^2-20k+1)$ with $\tau = 160k^2-55k+3$ and $\theta = 20k(8k-1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 5k-1$ and $\lambda_3 = -(40k-4)$ with $m_2 = 8(90k^2-20k+1)$ and $m_3 = 90k^2-20k+1$;

\begin{equation}
G \text{ is a strongly regular graph of order } n = 10(9k-1)^2 \text{ and degree } r = 5(90k^2-20k+1) \text{ with } \tau = 10k(25k-4) \text{ and } \theta = 5(5k-1)(10k-1), \text{ where } k \in \mathbb{N}.
\end{equation}

Its eigenvalues are $\lambda_2 = 40k-5$ and $\lambda_3 = -5k$ with $m_2 = 90k^2-20k+1$ and $m_3 = 8(90k^2-20k+1)$;

\begin{equation}
G \text{ is a strongly regular graph of order } n = 10(9k+1)^2 \text{ and degree } r = 4(90k^2+20k+1) \text{ with } \tau = 160k^2+55k+3 \text{ and } \theta = 20k(8k+1), \text{ where } k \in \mathbb{N}.
\end{equation}

Its eigenvalues are $\lambda_2 = 40k+4$ and $\lambda_3 = -(5k+1)$ with $m_2 = 90k^2+20k+1$ and $m_3 = 8(90k^2+20k+1)$;

\begin{equation}
G \text{ is a strongly regular graph of order } n = 10(9k+1)^2 \text{ and degree } r = 5(90k^2+20k+1) \text{ with } \tau = 10k(25k+4) \text{ and } \theta = 5(5k+1)(10k+1), \text{ where } k \in \mathbb{N}.
\end{equation}

Its eigenvalues are $\lambda_2 = 5k$ and $\lambda_3 = -(40k+5)$ with $m_2 = 8(90k^2+20k+1)$ and $m_3 = 90k^2+20k+1$;

\begin{equation}
G \text{ is a strongly regular graph of order } n = 28(9k-1)^2 \text{ and degree } r = 2(252k^2-56k+3) \text{ with } \tau = 112k^2-63k+5 \text{ and } \theta = 14k(8k-1), \text{ where } k \in \mathbb{N}.
\end{equation}

Its eigenvalues are $\lambda_2 = 7k-1$ and $\lambda_3 = -(56k-6)$ with $m_2 = 8(252k^2-56k+3)$ and $m_3 = 252k^2-56k+3$;

\begin{equation}
G \text{ is a strongly regular graph of order } n = 28(9k-1)^2 \text{ and degree } r = 7(252k^2-56k+3) \text{ with } \tau = 14(7k-1)(14k-1) \text{ and } \theta = 7(7k-1)(28k-3), \text{ where } k \in \mathbb{N}.
\end{equation}

Its eigenvalues are $\lambda_2 = 56k-7$ and $\lambda_3 = -7k$ with $m_2 = 252k^2-56k+3$ and $m_3 = 8(252k^2-56k+3)$;

\begin{equation}
G \text{ is a strongly regular graph of order } n = 28(9k+1)^2 \text{ and degree } r = 2(252k^2+56k+3) \text{ with } \tau = 112k^2+63k+5 \text{ and } \theta = 14k(8k+1), \text{ where } k \in \mathbb{N}.
\end{equation}

Its eigenvalues are $\lambda_2 = 56k+6$ and $\lambda_3 = -(7k+1)$ with $m_2 = 252k^2+56k+3$ and $m_3 = 8(252k^2+56k+3)$;

\begin{equation}
G \text{ is a strongly regular graph of order } n = 28(9k+1)^2 \text{ and degree } r = 7(252k^2+56k+3) \text{ with } \tau = 14(7k+1)(14k+1) \text{ and } \theta = 7(7k+1)(28k+3), \text{ where } k \in \mathbb{N}.
\end{equation}

Its eigenvalues are $\lambda_2 = 7k$ and $\lambda_3 = -(56k+7)$ with $m_2 = 8(252k^2+56k+3)$ and $m_3 = 252k^2+56k+3$.

\textbf{Proof.} Firstly, according to Remark 2.4 we have $\alpha\beta = 8(\alpha - 1)$, from which we find that $\alpha = 2, \beta = 5$ or $\alpha = 4, \beta = 7$ or $\alpha = 8, \beta = 8$. In view of this\footnote{The all results presented in this work are verified by using a computer program 	exttt{srgpar.exe}, which has been written by the author in the programming language Borland C++ Builder 5.5.} we have

\footnote{One can use the web page \url{https://www.win.tue.nl/˜aeb/graphs/srg/srgtab.html} that contains the parameters of strongly regular graphs from 5 upto 1300 vertices.}
obtain the strongly regular graphs represented in Theorem 2.4 \((1^0), (2^0), (3^0)\). Next, according to Proposition 2.7 it turns out that \(G\) belongs to the class \((4^0)\) or \((5^0)\) or \((6^0)\) or \((7^0)\) or \((8^0)\) or \((9^0)\) or \((10^0)\) if \(m_2 = 8m_3\). According to Proposition 2.8 it turns out that \(G\) belongs to the class \((4^0)\) or \((5^0)\) or \((6^0)\) or \((7^0)\) or \((8^0)\) or \((9^0)\) or \((10^0)\) or \((11^0)\) if \(m_3 = 8m_2\). □

Remark 2.13. We note that for some values of the parameter \(k\) it is possible that there exist no strongly regular graph. In view of this, we in this work describe the all feasible parameters \(n, r, \tau\) and \(\theta\) for strongly regular graphs with \(m_2 = qm_3\) and \(m_3 = qm_2\) for \(q = 5, 6, 7, 8\).

**REFERENCES**


[2] R. J. Elzinga, *Strongly regular graphs: values of \(\lambda\) and \(\mu\) for which there are only finitely many feasible \((v, k, \lambda, \mu)\)*. Electronic Journal of Linear Algebra ISSN 1081-3810, A publication of the International Linear Algebra Society, Volume 10, pp. 232-239, October 2003.


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STATISTICAL RELATIVE UNIFORM CONVERGENCE IN DUALLY RESIDUATED LATTICE TOTALLY ORDERED SEMIGROUPS

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ABSTRACT. We define the notions of statistical relative uniform convergence and statistical relative uniform Cauchy in dually residuated lattice totally ordered semigroups (simply, DRlt-semigroups). Then, we give some basic properties for statistically relatively uniform convergent sequences. Also, we introduce statistical relative uniform limit points and cluster points in DRlt-semigroups, then the relations between these and limit points of the sequence are given.

1. INTRODUCTION AND PRELIMINARIES

The notion of a Dually Residuated Lattice Ordered Semigroup (simply, a DRI-semigroup), a broad generalization of Brouwerian Algebras and commutative l-groups, has been introduced and studied by Swamy [9–11]. Then, Jasem [6] introduced u-uniform convergence and relatively uniform convergence for DRI-semigroups. In the current work, we introduce the notion of statistical relative uniform convergence and also the definition of statistical relative uniform Cauchy in DRlt-semigroups is given. Then, we give some basic properties for statistically relatively uniform convergent sequences.

We now recall the notion of a DRI-semigroup that has been introduced by Swamy in [9] and related properties used in the paper.

A system $A = (A, +, \leq, -)$ is called a dually residuated lattice ordered semigroup (simply, a DRI-semigroup) if and only if

1. $(A, +, \leq)$ is a commutative lattice ordered semigroup with zero element $0$, i.e. $(A, +)$ is a commutative semigroup with zero $0$ and $(A, \leq)$ is a lattice such that $a + (b \lor c) = (a + b) \lor (a + c)$ and $a + (b \land c) = (a + b) \land (a + c)$ for all $a, b, c \in A$,
2. given $a, b$ in $A$ there exists a least $x$ in $A$ such that $b + x \geq a$, and we denote this $x$ by $a - b$ (for a given pair $a, b$ this $x$ is uniquely determined),
3. $(a - b) \lor 0 + b \leq a \lor b$ for all $a, b \in A$.

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(4) \((a - a) \geq 0\).

The following theorem shows that any DRl-semigroup can be equationally defined:

**Theorem 1.1.** [9] Any DRl-semigroup can be equationally defined as an algebra with binary operations +, ∨, ∧, −, by replacing (2) by the equations:

\[ x + (y - x) \geq y, \; x - y \leq (x \lor z) - y, \; (x + y) - y \leq x. \]

Any abelian lattice ordered group is a DRl-semigroup. For any \(a\) and \(b\) in a DRl-semigroup \(A\), we shall write \(|a - b| = (a - b) \lor (b - a)\), \(|a - b|\) is called the symmetric difference of \(a\) and \(b\). The symmetric difference satisfies the following conditions:

1. \(|a - b| \geq 0\), \(|a - b| = 0\) if and only if \(a = b\),
2. \(|a - b| = |b - a|\),
3. \(|a - c| \leq |a - b| + |b - c|\).

Any DRl-semigroup is an autometrized algebra with the symmetric difference ([9], Theorem 9).

**Theorem 1.2.** [6] Let \(A\) be a DRl-semigroup, \(a, b, c, d \in A\). Then

\[
\begin{align*}
(i) \quad & (a - b) + (c - d) \geq (a + c) - (b + d), \\
(ii) \quad & |a - b| + |c - d| \geq |(a + c) - (b + d)|, \\
(iii) \quad & (a - b) + (c - d) \geq (c - b) - (d - a).
\end{align*}
\]

In this paper, we will need the following assumptions in a DRl-semigroup \(A\) from [9]:

Let \(a, b, c \in A\). Then \((A1)\) \(a \leq b\) implies \(a - c \leq b - c\) and \(c - b \leq c - a\),\n\((A2)\) \((a \lor b) - c = (a - c) \lor (b - c)\),\n\((A3)\) \(a - (b \land c) = (a - b) \lor (a - c)\),\n\((A4)\) \(a - (b + c) = (a - b) - c = (a - c) - b\),\n\((A5)\) \((a - b) + (b - c) \geq (a - c)\),\n\((A6)\) \(a - (b - c) \leq (a - b) + c\) and \((a + b) - c \leq (a - c) + b\).

We denote \(A^+ = \{x \in A; x \geq 0\}\). A DRl-semigroup is said to be Archimedean if for each \(x, y \in A^+\), \(nx \leq y\) for each \(n \in \mathbb{N}\) implies \(x = 0\).

2. **Statistical Relative Uniform Convergence**

Jasem [6] introduced notions of a u-uniform convergence and a relatively uniform convergence in DRl-semigroup as follows.

**Definition 2.1.** Let \(A\) be a DRl-semigroup, \((x_k)\) a sequence in \(A\), \(u \in A^+\). It is said that a sequence \((x_k)\) in \(A\) converges u-uniformly to an element \(x \in A\), written \(x_k \xrightarrow{u} x\), if the following condition is satisfied:

for each \(p \in \mathbb{N}\) there exists \(k_p \in \mathbb{N}\), such that \(p |x_k - x| \leq u\) for each \(k \in \mathbb{N}, k \geq k_p\).
Definition 2.2. Let \( A \) be a DRI-semigroup. We say that a sequence \((x_k)\) in \( A \) relatively uniformly converges (briefly ru-converges) to an element \( x \in A \), in symbols \( x_k \rightarrow x \), whenever there exists \( u \in A^+ \) such that \( x_k \rightarrow^u x \).

Example 2.1. Let \( A \) be the unit interval \([0, 1]\) of real numbers, \( a \oplus b = \min\{1, a + b\} \). With the usual ordering \((A, \oplus, \leq, -)\) is a DRI-semigroup. Let define \((x_k)\) by
\[
x_k = \begin{cases} 
\frac{1}{2}, & k = p^2 \\
\frac{p^2}{2p^2 + 2}, & k \neq p^2, \quad p = 1, 2, 3, \ldots
\end{cases}
\]
and \( u \in \left[\frac{1}{4}, 1\right] \). Then
\[
p \left| x_k - \frac{1}{2} \right| = p \left[ \left( x_k - \frac{1}{2} \right) \lor \left( \frac{1}{2} - x_k \right) \right] \\
= p \left( \frac{1}{2} - x_k \right) \\
= \left( \frac{1}{2} - x_k \right) \oplus \ldots \oplus \left( \frac{1}{2} - x_k \right) \quad p \text{ times}
\]
and there exists \( k_p \in \mathbb{N} \), such that \( p \left| x_k - \frac{1}{2} \right| \leq u \) for each \( k \in \mathbb{N}, k \geq k_p \). Hence, \( x_k \rightarrow \frac{1}{2} \).

The statistical convergence was first introduced by Steinhaus [8] and after the papers of Connor [1] and Fridy [3] about this convergence method, the developments started and it was studied by other authors [2, 5, 7]. We use \( \mathbb{N} \) for the set of all positive integers. Let \( K \subseteq \mathbb{N} \), then the natural density of \( K \) denoted by \( \delta(K) \), is given by
\[
\delta(K) := \lim_{n \to \infty} \frac{1}{n} \left| \{k \leq n : k \in K\} \right|
\]
whenever the limit exists, where \( |B| \) denotes the cardinality of the set \( B \). It is known that the density may not exists for each set \( K \). But the upper density \( \overline{\delta} \) always exists and it is identified by \( \overline{\delta}(K) := \limsup_n \frac{1}{n} \left| \{k \leq n : k \in K\} \right| \). Moreover, \( \overline{\delta}(K) > 0 \).

Now, we introduce the notion of the statistical relative uniform convergence with respect to the dually residuated lattice totally ordered semigroup \( A \) (simply, a DRLt-semigroup \( A \)) which is the DRI-semigroup such that \( A \) is totally ordered set.

Definition 2.3. Let \( A \) be a DRLt-semigroup, \((x_k)\) a sequence in \( A \), \( u \in A^+ \). It is said that a sequence \((x_k)\) in \( A \) is statistically \( u \)-uniform convergent to \( x \in A \) provided that for each \( p \in \mathbb{N} \)
\[
\delta\left(\{k \in \mathbb{N} : p \left| x_k - x \right| \geq u\}\right) = 0
\]
or equivalently
\[
\lim_{n \to \infty} \frac{1}{n} \left| \{k \leq n : p \left| x_k - x \right| \geq u\} \right| = 0.
\]
In that case, we write \( x_k \xrightarrow{\text{st u}} x \).
**Definition 2.4.** Let $A$ be a DRlt-semigroup. We say that a sequence $(x_k)$ in $A$ is statistically relatively uniform convergent to $x \in A$ whenever there exists $u \in A^+$ such that $x_k \xrightarrow{st^u} x$. In that case, we write $x \xrightarrow{st^u} x$ and $x$ is said to be $st^u$-limit.

**Example 2.2.** Let $A$ be the unit interval $[0, 1]$ of real numbers, $a \oplus b = \min\{1, a + b\}$. With the usual ordering $(A, \oplus, \leq, -)$ is a DRlt-semigroup. Let define $(x_k)$ by

$$x_k = \begin{cases} 1, & k = n^2 \\ 0, & k \neq n^2 \end{cases}, \quad n = 1, 2, 3,....$$

and $u \neq 0$. Then

$$p |x_k - 0| = p |x_k| = px_k = x_k \oplus \ldots \oplus x_k = \underbrace{x_k \oplus \ldots \oplus x_k}_{p\text{ times}}.$$ 

Hence, $\delta(\{k \in \mathbb{N} : p |x_k - 0| \geq u\}) = 0$, and we get $x_k \xrightarrow{st^u} 0$. However, $\{k \in \mathbb{N} : p |x_k - 0| \geq u\}$ is an infinite set. So, $x_k \not\rightarrow 0$.

**Theorem 2.1.** Let $A$ be an Archimedean DRlt-semigroup, $(x_k)$ a sequence in $A$, $x, y \in A$. Then $x \xrightarrow{st^u} x \xrightarrow{st^u} y$. Then there exist $u, v \in A^+$ such that $x_k \xrightarrow{st^u} x, x_k \xrightarrow{st^u} y$. Let $p \in \mathbb{N}$ and define the following sets:

$$K_1(p, u) = \{k \in \mathbb{N} : p |x_k - x| \geq u\},$$

$$K_2(p, v) = \{k \in \mathbb{N} : p |x_k - y| \geq v\}.$$ 

Since $x_k \xrightarrow{st^u} x$, $\delta(K_1(p, u)) = 0$. Furthermore, using $x_k \xrightarrow{st^u} y$, we get $\delta(K_2(p, v)) = 0$. Now let $K(p, u + v) := K_1(p, u) \cap K_2(p, v)$. Then we observe that $\delta(K(p, u + v)) = 0$ which implies that $\delta(\mathbb{N}\setminus K(p, u + v)) = 1$. If $k \in \mathbb{N}\setminus K(p, u + v)$, then we have

$$u + v \geq p |x_k - x| + p |x_k - y| = p (|x_k - x| + |x_k - y|) = p (|x - x_k| + |x_k - y|) \geq p |x - y|.$$ 

So Archimedeanicity of $A$ implies $|x - y| = 0$. Hence $x = y$. \hfill \Box

**Theorem 2.2.** Let $A$ be an Archimedean DRlt-semigroup and $(x_k)$ be a sequence in $A$. If $(x_k)$ ru-converges to $x$, then $x \xrightarrow{st^u} x$.

**Proof.** By hypothesis, for each $p \in \mathbb{N}$ there exists $k_p \in \mathbb{N}$, such that $p |x_k - x| \leq u$ for all $k \geq k_p$. From this, it can be said that the set $\{k \in \mathbb{N} : p |x_k - x| \geq u\}$ has at most finitely many terms. It is known that the density of every finite subset of the natural numbers is zero, hence we can see that $\delta(\{k \in \mathbb{N} : p |x_k - x| \geq u\}) = 0$, whence the result. \hfill \Box
**Theorem 2.3.** Let $A$ be an Archimedean DRlt-semigroup, $(x_k)$ and $(y_k)$ be sequences in $A$, $x, y \in A$. Let $st^r - \lim x_k = x$ and $st^r - \lim y_k = y$. Then

(i) $st^r - \lim (x_k + y_k) = x + y,$
(ii) $st^r - \lim (x_k - y_k) = x - y,$
(iii) $st^r - \lim (x_k \lor y_k) = x \lor y,$
(iv) $st^r - \lim (x_k \land y_k) = x \land y.$

**Proof.** Let $st^r - \lim x_k = x$ and $st^r - \lim y_k = y$. Then there exist $u, v \in A^+$, such that $x_k \stackrel{st^r}{\to} x, y_k \stackrel{st^r}{\to} y$. Now let $p \in \mathbb{N}$ and define the following sets:

$$K_1(p, u) := \{ k \in \mathbb{N} : p|x_k - x| \geq u \},$$
$$K_2(p, v) := \{ k \in \mathbb{N} : p|y_k - y| \geq v \}.$$

Since $x_k \stackrel{st^r}{\to} x$ and $y_k \stackrel{st^r}{\to} y$, we get $\delta(K_1(p, u)) = 0$ and $\delta(K_2(p, v)) = 0$. Now let $K(p, u + v) := K_1(p, u) \cap K_2(p, v)$. Then we observe that $\delta(K(p, u + v)) = 0$ which implies that $\delta(N(K(p, u + v))) = 1$.

(i) If $k \in \mathbb{N} \setminus K(p, u + v)$, then in view of Theorem 1.2 (ii), we have

$$u + v \geq p|x_k - x| + p|y_k - y| = p(|x_k - x| + |y_k - y|) \geq p|(x_k + y_k) - (x + y)|.$$

This shows that

$$\delta(\{ k \in \mathbb{N} : p|(x_k + y_k) - (x + y)| \geq u + v \}) = 0.$$

So $x_k + y_k \stackrel{st^r \lor}{\to} x + y$ and hence $st^r - \lim (x_k + y_k) = x + y.$

(ii) Similar to (i), we can prove that $st^r - \lim (x_k - y_k) = x - y.$

(iii) If $k \in \mathbb{N} \setminus K(p, u + v)$, then according to (A2) we obtain

$$u + v \geq p|x_k - x| + p|y_k - y| = p(|x_k - x| + |y_k - y|) \geq p(|x_k - x| \lor |y_k - y|) = p[(x_k - x) \lor (x - x_k) \lor (y_k - y) \lor (y - y_k)]$$

$$\geq p[(x_k - x) \lor (y_k - y) \lor (x - x_k) \lor (y - y_k)]$$

$$\geq p[[((x_k \lor y_k) - (x \lor y)) \lor ((y \lor x) - (x_k \lor y_k)) \lor ((y \lor x) - (x_k \lor y_k))]] = p[(x \lor y) - (x_k \lor y_k)].$$

Then,

$$\delta(\{ k \in \mathbb{N} : p|(x_k \lor y_k) - (x \lor y)| \geq u + v \}) = 0.$$

So $x_k \lor y_k \stackrel{st^r \lor}{\to} x \lor y$ and we have $st^r - \lim (x_k \lor y_k) = x \lor y.$

(iv) If $k \in \mathbb{N} \setminus K(p, u + v)$, then in view of (A3) we get

$$u + v \geq p|x_k - x| + p|y_k - y| = p(|x_k - x| + |y_k - y|) \geq p(|x_k - x| \lor |y_k - y|) = p[(x_k - x) \lor (x - x_k) \lor (y_k - y) \lor (y - y_k)]$$

$$\geq p[(x_k - x) \lor (y_k - y) \lor (x - x_k) \lor (y - y_k)]$$

$$\geq p[(((x_k \land y_k) - x) \lor ((x_k \land y_k) - y) \lor ((x \land y) - x_k) \lor ((x \land y) - y_k)]$$
Theorem 2.5. Proof. Since \( zk \) sequences in \( A \), By hypothesis, since \( x_k \) sequences in \( A \), we obtain \( st^r - \lim x_k = x \), \( st^r - \lim y_k = y \), then \( x \leq y \).

Theorem 2.4. Let \( A \) be an Archimedean DRlt-semigroup, \( (x_k) \) and \( (y_k) \) be sequences in \( A \). If \( x_k \leq y_k \) for all \( k \in K \subseteq \mathbb{N} \) with \( \delta(K) = 1 \) and \( st^r - \lim x_k = x \), \( st^r - \lim y_k = y \), then \( x \leq y \).

Proof. By hypothesis, since \( st^r - \lim x_k = x \) and \( st^r - \lim y_k = y \) we get from Theorem 2.3 (iii) that \( st^r - \lim (x_k \lor y_k) = x \lor y \). Therefore, there exists \( u \in A^+ \) such that \( x_k \lor y_k \rightharpoonup x \lor y \). Let \( p \in \mathbb{N} \) and define the following set:

\[
K_1(p,u) := \{ k \in \mathbb{N} : p \mid (x_k \lor y_k) - (x \lor y) \mid \geq u \}.
\]

Since \( x_k \lor y_k \rightharpoonup x \lor y \), \( \delta(K_1(p,u)) = 0 \). This implies that

\[
\delta(\mathbb{N} \setminus K_1(p,u)) = \delta(\{ k \in \mathbb{N} : u \geq p \mid (x_k \lor y_k) - (x \lor y) \mid \}) = 1.
\]

Let \( k \in K \cap (\mathbb{N} \setminus K_1(p,u)) \). It is clear that \( \delta(K \cap (\mathbb{N} \setminus K_1(p,u))) = 1 \) and \( u \geq p \mid (x_k \lor y_k) - (x \lor y) \mid = p \mid y_k - (x \lor y) \mid \).

So \( \delta(\{ k \in \mathbb{N} : p \mid y_k - (x \lor y) \mid \geq u \}) = 0 \). Then from this, we have that \( st^r - \lim y_k = x \lor y \). Because of \( st^r - \lim y_k = x \lor y \), we get \( x \leq y \). 

Theorem 2.5. Let \( A \) be an Archimedean DRlt-semigroup, \( (x_k) \), \( (y_k) \) and \( (z_k) \) be sequences in \( A \). If

(i) \( x_k \leq y_k \leq z_k \) for all \( k \in K \subseteq \mathbb{N} \) with \( \delta(K) = 1 \) and \( st^r - \lim x_k = x \), \( st^r - \lim z_k = x \),

then \( st^r - \lim y_k = x \).

Proof. Since \( st^r - \lim x_k = st^r - \lim z_k = x \), there exist \( u, v \in A^+ \), such that \( x_k \rightharpoonup x \), \( z_k \rightharpoonup x \). Let \( p \in \mathbb{N} \) and define the following sets:

\[
K_1(p,u) := \{ k \in \mathbb{N} : p \mid x_k - x \mid \geq u \},
\]

\[
K_2(p,v) := \{ k \in \mathbb{N} : p \mid z_k - x \mid \geq v \},
\]

then \( \delta(K_1(p,u)) = 0 \) and \( \delta(K_2(p,v)) = 0 \). Let \( k \in K \cap (\mathbb{N} \setminus K_1(p,u)) \cap (\mathbb{N} \setminus K_2(p,v)) \).

It is clear that \( \delta(K \cap (\mathbb{N} \setminus K_1(p,u)) \cap (\mathbb{N} \setminus K_2(p,v))) = 1 \) and

\[
\begin{align*}
  u + v & \geq p \mid x_k - x \mid + p \mid z_k - x \mid = p \mid (x_k - x) + (z_k - x) \mid \\
  & \geq p \mid (x_k - x) \lor (z_k - x) \mid \\
  & = p \mid (x_k - x) \lor (x - x_k) \lor (z_k - x) \lor (x - z_k) \mid \\
  & \geq p \mid (z_k - x) \lor (x - x_k) \mid.
\end{align*}
\]

In view of (A1) from \( x_k \leq y_k \leq z_k \), we get \( y_k - x \leq z_k - x \), \( x - y_k \leq x - x_k \). This yields \( (y_k - x) \lor (x - y_k) \leq (z_k - x) \lor (x - x_k) \). Therefore \( u + v \geq p \mid (z_k - x) \lor (x - x_k) \mid \geq p \mid (y_k - x) \lor (x - y_k) \mid = p \mid y_k - x \mid \).

Let

\[
K_3(p,u + v) := \{ k \in \mathbb{N} : p \mid y_k - x \mid \geq u + v \}.
\]
It is clear that the set
\[ K_3(p,u+v) \subseteq K_1(p,u) \cup K_2(p,v) \cup (\mathbb{N}\setminus K) \]
and \( \delta(K_3(p,u+v)) = 0. \) Hence \( st^r - \lim y_k = x. \)

We now introduce the notion of \( st^r \)-Cauchy sequence and give a characterization.

**Definition 2.5.** Let \( A \) be an Archimedean DRlt-semigroup and \( (x_k) \) be a sequence in \( A. \) We say that a sequence \( (x_k) \) is a \( st^r \)-Cauchy with respect to the statistical relative uniform convergence, if for some \( u \in A^+ \) and each \( p \in \mathbb{N}, \) there exists a positive integer \( m \in \mathbb{N} \) satisfying
\[ \delta(\{k \in \mathbb{N} : p|x_k - x_m| \geq u\}) = 0. \]

**Theorem 2.6.** Let \( A \) be an Archimedean DRlt-semigroup and \( (x_k) \) be a sequence in \( A. \) If \( (x_k) \) is statistically relatively uniform convergent, then it is a \( st^r \)-Cauchy.

**Proof.** Assume that \( st^r - \lim x_k = x. \) Then there exists \( u \in A^+, \) such that \( x_k \xrightarrow{st^u} x. \) Therefore the set \( K_1(p,u) := \{k \in \mathbb{N} : p|x_k - x| \geq u\} \) has density zero. This implies that the set \( \mathbb{N}\setminus K_1(p,u) \) has density one and therefore non empty. So we can choose \( m \in \mathbb{N} \) with \( m \notin K_1(p,u), \) but then we have \( u \geq p|x_m - x| \). Let \( K(p,2u) := \{k \in \mathbb{N} : p|x_k - x_m| \geq 2u\}. \) We prove that \( \delta(K(p,2u)) = 0. \) Since
\[ 2u \geq p|x_k - x| + p|x_m - x| = p(|x_k - x| + |x - x_m|) \]
\[ \geq p|x_k - x_m| \]
we can write
\[ \{k \in \mathbb{N} : 2u \geq p|x_k - x_m|\} \supset \{k \in \mathbb{N} : u \geq p|x_k - x|\} \cap \{k \in \mathbb{N} : u \geq p|x_m - x|\}. \]

From this we get
\[ \delta(\{k \in \mathbb{N} : p|x_k - x_m| \geq 2u\}) \leq \delta(\{k \in \mathbb{N} : p|x_k - x| \geq u\}) + \delta(\{k \in \mathbb{N} : p|x_m - x| \geq u\}). \]

Hence
\[ \delta(\{k \in \mathbb{N} : p|x_k - x_m| \geq 2u\}) = 0 \]
which shows that our claim \( \delta(K(p,2u)) = 0 \) holds true. \( \square \)

3. Limit Points and Cluster Points in DRlt-semigroups

Fridy defined statistical limit points and statistical cluster points of a number sequence \( (x_k) \) in 1993 [4]. Now we study the concepts of statistical relative uniform limit points and cluster points in DRlt-semigroups. Also, we give relations between them and the set of limit points in DRlt-semigroups.

**Definition 3.1.** Let \( A \) be a DRlt-semigroup and \( (x_k) \) be a sequence in \( A, u \in A^+. \) \( L \in A \) is said to be a limit point of the sequence \( (x_k) \) providing that there is a subsequence of \( (x_k) \) that ru-converges to \( x. \) Let the set of all limit points of the sequence denoted by \( L_{ru}[(x_k)]. \)
Definition 3.2. Let $A$ be a DRlt-semigroup and $(x_k)$ be a sequence in $A$, $u \in A^+$. If \( \{x_k(n)\} \) is a subsequence of $(x_k)$ and $K := \{k(n) \in \mathbb{N} : n \in \mathbb{N}\}$ then $\{x\}^*_K$ in which case $\delta(K) = 0$ is the abbreviation of $\{x_k(n)\}$ and $\{x\}^*_K$ is said to be thin subsequence or a subsequence of density zero. In other words, if $K$ does not have density zero then $\{x\}^*_K$ is a nonthin subsequence of $(x_k)$.

Definition 3.3. Let $A$ be a DRlt-semigroup and $(x_k)$ be a sequence in $A$, $u \in A^+$. $x \in A$ is said to be a statistical relative uniform limit point of the sequence $(x_k)$ on the condition that there is a nonthin subsequence of $(x_k)$ that ru-converges to $x$. Then we say $x$ is a st’—limit point of sequence $(x_k)$. Let $\Lambda_{sr}^*[(x_k)]$ denote the set of all st’—limit points of the sequence.

Example 3.1. Let $A$ and $(x_k)$ be the same as in Example 2.2, then $L_{ru}^*[(x_k)] = \{0, 1\}$ and $\Lambda_{sr}^*[(x_k)] = \{0\}$.

It is clear that $\Lambda_{sr}^*[(x_k)] \subset L_{ru}^*[(x_k)]$ for any sequence $(x_k)$. To show that $\Lambda_{sr}^*[(x_k)]$ and $L_{ru}^*[(x_k)]$ can be very different, we give an example as follows.

Example 3.2. Let $A$ be the unit interval $[0, 1]$ of real numbers, $a \oplus b = \min\{1, a + b\}$. With the usual ordering $(A, \oplus, \leq, -)$ is a DRlt-semigroup. Let $(r_k)$ be a sequence whose range is the set of all rational numbers of $[0, 1]$ and define $(x_k)$ by

\[
x_k = \begin{cases} 
    r_j, & k = j^2, \\
    \frac{1}{3}, & k \neq j^2 \text{ and } k \text{ is odd} \\
    \frac{1}{2}, & k \neq j^2 \text{ and } k \text{ is even}
\end{cases}, \quad j = 1, 2, 3, \ldots,
\]

Then for $k \neq j^2$ and $k$ is odd, $p|x_k - x| = p|\frac{1}{3} - \frac{1}{2}| = 0 \leq u$. Therefore $x_k \to \frac{1}{2}$.

For $k \neq j^2$ and $k$ is even, $p|x_k - x| = p|\frac{1}{3} - \frac{1}{5}| = 0 \leq u$. Hence $x_k \to \frac{1}{3}$. We get $\Lambda_{sr}^*[(x_k)] = \{\frac{1}{2}, \frac{1}{3}\}$. However, for every $x \in [0, 1]$ and $k = j^2$,

\[
p|x_k - x| = p|r_j - x| = |r_j - x| \oplus \ldots \oplus |r_j - x| = \min\{1, |r_j - x| + \ldots + |r_j - x|\} \leq 1
\]

\( \{r_k : k \in \mathbb{N}\} \) is dense in $[0, 1]$ implies that $L_{ru}^*[(x_k)] = [0, 1]$.

Definition 3.4. Let $A$ be a DRlt-semigroup and $(x_k)$ be a sequence in $A$, $u \in A^+$. Then $x \in A$ is said to be a statistical relative uniform cluster point of the sequence $(x_k)$ provided that for each $p \in \mathbb{N},$

\[
\delta\left(\{k \in \mathbb{N} : p|x_k - x| \leq u\}\right) > 0.
\]

In that case we say that $x$ is an st’—cluster point of sequence $(x_k)$. Let $\Gamma_{sr}^*[(x_k)]$ denote the set of all st’—cluster points of sequence $(x_k)$.

Theorem 3.1. Let $A$ be an Archimedean DRlt-semigroup, $(x_k)$ be a sequence in $A$, $x \in A$. For any sequence $(x_k)$, $\Lambda_{sr}^*[(x_k)] \subset \Gamma_{sr}^*[(x_k)]$. 

Proof. Suppose \( x \in \Lambda_{sr}[\{x_k\}] \). So, there is a nonthin subsequence \( \{x_{k(n)}\} \) of \( (x_k) \) that ru-converges to \( x \), i.e.

\[
\delta \left( \{ k(n) \in \mathbb{N} : p \left| x_{k(n)} - x \right| \leq u \} \right) = d > 0.
\]

Since

\[
\{ k \in \mathbb{N} : p \left| x_k - x \right| \leq u \} \supset \{ k(n) \in \mathbb{N} : p \left| x_{k(n)} - x \right| \leq u \}
\]

for each \( p \in \mathbb{N} \), we have

\[
\{ k \in \mathbb{N} : p \left| x_k - x \right| \leq u \} \supset \{ k(n) \in \mathbb{N} : n \in \mathbb{N} \} \setminus \{ k(n) \in \mathbb{N} : p \left| x_{k(n)} - x \right| \geq u \}.
\]

Since \( \{x_{k(n)}\} \) ru-converges to \( x \), the set

\[
\{ k(n) \in \mathbb{N} : p \left| x_{k(n)} - x \right| \geq u \}
\]

is finite for any \( p \in \mathbb{N} \). Therefore,

\[
\delta((\{ k \in \mathbb{N} : p \left| x_k - x \right| \leq u \})) \geq \delta((\{ k(n) \in \mathbb{N} : n \in \mathbb{N} \})) - \delta((\{ k(n) \in \mathbb{N} : p \left| x_{k(n)} - x \right| \geq u \})).
\]

Hence

\[
\delta((\{ k \in \mathbb{N} : p \left| x_k - x \right| \leq u \})) > 0
\]

that means \( x \in \Gamma_{sr}[\{x_k\}] \). \(\square\)

Example 3.3. Define the sequence \( (x_k) \) by

\[
x_k = \frac{1}{m}, k = 2^{m-1}(2q+1),
\]

i.e., \( m-1 \) is the number of factors of 2 in the prime factorization of \( k \). For each \( p \in \mathbb{N}, u = 1, \)

\[
px_k = \underbrace{x_k \oplus \ldots \oplus x_k}_{p \text{ times}} = \frac{1}{m} \oplus \ldots \oplus \frac{1}{m} \leq 1,
\]

it is easy to see that for each \( m, \delta((\{ k \in \mathbb{N} : px_k \leq 1 \})) = 2^{-m} > 0, \) whence \( \frac{1}{m} \in \Lambda_{sr}[\{x_k\}] \). Also, \( \delta((\{ k \in \mathbb{N} : p \left| x_k - 0 \right| \leq 1 \}) = \delta((\{ k \in \mathbb{N} : 0 \leq px_k \leq 1 \})) = 2^{-m}, \) so \( 0 \in \Gamma_{sr}[\{x_k\}] \) and we have \( \Gamma_{sr}[\{x_k\}] = \{0\} \cup \{ \frac{1}{m} : m = 1, 2, \ldots \} \). Now we claim that \( 0 \notin \Lambda_{sr}[\{x_k\}] ; \) for, if \( \{x_k\} \) is a subsequence that has limit zero, then we obtain that \( \delta(K) = 0 \) by observing that for each \( m, \) there exists \( M > 0 \) such that

\[
|K_n| = |\{ k \in K_n : px_k \geq 1 \}| + |\{ k \in K_n : px_k \leq 1 \}|
\]

\[
\leq M + |\{ k \in \mathbb{N} : px_k \leq 1 \}|
\]

\[
\leq M + \frac{1}{2^m}.
\]

Hence \( \delta(K) \leq 2^{-m} \), and since \( m \) is arbitrary this implies that \( \delta(K) = 0 \).

Theorem 3.2. Let \( A \) be an Archimedean DRLt-semigroup, \( (x_k) \) be a sequence in \( A \), \( x \in A \). For any sequence \( (x_k) \), \( \Gamma_{sr}[\{x_k\}] \subset L_{sr}[\{x_k\}] \).

Proof. \( x \in \Gamma_{sr}[\{x_k\}] \), then

\[
\delta((\{ k \in \mathbb{N} : p \left| x_k - x \right| \leq u \})) > 0
\]
for each $p \in \mathbb{N}$. We set $\{x\}_K$ a nonthin subsequence of $(x_k)$ such that

$$K = K(p,u) := \{ k(n) \in \mathbb{N} : p|x_{k(n)} - x| \geq u \}$$

for each $p \in \mathbb{N}$ and $\delta(K) \neq 0$. Hence there are infinitely many elements in $K$, $x \in L_{st'}[(x_k)]$. □

**Theorem 3.3.** Let $A$ be an Archimedean DRlt-semigroup, $(x_k)$ be a sequence in $A$. For a sequence $(x_k)$, $st' - \lim x_k = x_0$ then $\Lambda_{st'}[(x_k)] = \Gamma_{st'}[(x_k)] = \{x_0\}$.

**Proof.** Assume that $st' - \lim x_k = x_0$, then there exists $u \in A^+$ such that $x_k \to x_0$. First we show that $\Lambda_{st'}[(x_k)] = \{x_0\}$. We suppose that $\Lambda_{st'}[(x_k)] = \{x_0, y_0\}$ such that $p|x_0 - y_0| \geq 2u$. Hence, there exist $\{x_{k(n)}\}$ and $\{x_{j(i)}\}$ nonthin subsequences of $(x_k)$ that ru-converge to $x_0, y_0$, respectively. Since $\{x_{j(i)}\}$ ru-converge to $y_0$, for each $p \in \mathbb{N}$

$$K := K(p,u) := \{ j(i) \in \mathbb{N} : p|x_{j(i)} - y_0| \geq u \}$$

is a finite set so $\delta(K) = 0$. Then observe that

$$\{j(i) \in \mathbb{N} : i \in \mathbb{N}\} = \{j(i) \in \mathbb{N} : p|x_{j(i)} - y_0| \geq u\} \cup \{j(i) \in \mathbb{N} : p|x_{j(i)} - y_0| \leq u\}$$

which implies

$$\delta(\{j(i) \in \mathbb{N} : p|x_{j(i)} - y_0| \leq u\}) \neq 0. \quad (3.1)$$

Since $st' - \lim x_k = x_0$,

$$\delta(\{k \in \mathbb{N} : p|x_k - x_0| \geq u\}) = 0 \quad (3.2)$$

for each $p \in \mathbb{N}$. Therefore, we can write

$$\delta(\{k \in \mathbb{N} : p|x_k - x_0| \leq u\}) \neq 0.$$  

For every $p|x_0 - y_0| \geq 2u$,

$$\{j(i) \in \mathbb{N} : p|x_{j(i)} - y_0| \leq u\} \cap \{k \in \mathbb{N} : p|x_k - x_0| \leq u\} = \emptyset.$$  

Hence, we can write

$$\{j(i) \in \mathbb{N} : p|x_{j(i)} - y_0| \leq u\} \subseteq \{k \in \mathbb{N} : p|x_k - x_0| \leq u\}.$$  

Therefore

$$\overline{\delta}(\{j(i) \in \mathbb{N} : p|x_{j(i)} - y_0| \leq u\}) \leq \overline{\delta}(\{k \in \mathbb{N} : p|x_k - x_0| \geq u\}) = 0.$$  

This contradicts (3.1). Hence $\Lambda_{st'}[(x_k)] = \{x_0\}$.

Now we suppose that $\Gamma_{st'}[(x_k)] = \{x_0, z_0\}$ such that $p|x_0 - y_0| \geq 2u$. Then

$$\overline{\delta}(\{k \in \mathbb{N} : p|x_k - z_0| \leq u\}) \neq 0. \quad (3.3)$$

Since

$$\{k \in \mathbb{N} : p|x_k - x_0| \leq u\} \cap \{k \in \mathbb{N} : p|x_k - z_0| \leq u\} = \emptyset$$

for every $p|x_0 - y_0| \geq 2u$, so

$$\{k \in \mathbb{N} : p|x_k - x_0| \geq u\} \supset \{k \in \mathbb{N} : p|x_k - z_0| \leq u\}.$$
Therefore
\[ \bar{\delta}(\{ k \in \mathbb{N} : p|x_k - x_0| \geq u \}) \geq \bar{\delta}(\{ k \in \mathbb{N} : p|x_k - z_0| \leq u \}). \]  
(3.4)

From (3.3), the right hand-side of (3.4) is greater than zero and from (3.2), the left hand-side of (3.4) equals zero. This is a contradiction. Hence \( \Gamma_{str}[x_k] = \{ x_0 \}. \) □

REFERENCES

AN EQUIVALENT FORM OF THE PRIME NUMBER THEOREM

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ABSTRACT. A simple proof is given that \( \sum \frac{\mu(n)d(n)}{n} = 0 \) using the Prime Number Theorem. It is shown that this is equivalent to the estimate \( \sum_{n \leq x} \mu(n)d(n) = o(x) \) and to the Prime Number Theorem.

1. INTRODUCTION

Let \( \mu(n) \) denote the Moebius function and \( d(n) \) the divisor function. The question ” \( \sum \frac{\mu(n)d(n)}{n} = 0 ? \) ” was posed in ([4], p 1599). It can indeed be settled positively by applying the estimate in Ram Murty ([3], 4.4.6) \( \sum_{n \leq x} \mu(n)d(n) = o(x) \) (see Remark 1.6 below) W.Narkiewicz in a letter indicated that it can be proved by contour integrals. In this note we give one more proof (Prop 1.7). We also show (Prop 1.11) that this convergence is equivalent to the Prime Number Theorem.

Lemma 1.1.
\[
\prod_p (1 - \frac{2}{p^s}) = \sum_n \frac{\mu(n)d(n)}{n^s} \quad (Re \ s > 1)
\]

Proof. For any multiplicative function \( f(n) \) we have ([1], Theorem 11.7)
\[
\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \frac{f(p)}{p^s} + \cdots + \frac{f(p^k)}{p^{ks}} + \cdots\right)
\]
Take \( f(n) = \mu(n)d(n) \), \( f(p) = \mu(p)d(p) = -2 \) and \( f(p^k) = \mu(p^k)d(p^k) = 0 \) for \( k > 2 \). Hence the equality. \( \square \)

Lemma 1.2.
\[
\prod_p \left(1 - \frac{2}{p^s}\right) = \prod \{(1 - \frac{1}{p^s})^2\} \prod \left(1 - \frac{1}{p^{2s}(1 - \frac{1}{p})^2}\right) \quad (Re \ s > 1)
\]

Proof. ([3], 4.4.6) We verify the factorization.

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\[
\left(1 - \frac{2}{x}\right) = \left(1 - \frac{1}{x}\right)^2 \left(1 - \frac{1}{x^2 (1 - \frac{1}{x})^2}\right)
\]

and put \(x = p^s\), take product over all primes \(p\), and rearrange terms using absolute convergence for \(\text{Re} \, s > 1\).

\[
\text{RHS} = \left(1 - \frac{1}{x}\right)^2 \left(\frac{x^2 (1 - \frac{1}{x})^2 - 1}{x^2 (1 - \frac{1}{x})^2}\right)
\]

\[
= \left(1 - \frac{1}{x}\right)^2 \left(\frac{(x - 1)^2 - 1}{(x - 1)^2}\right)
\]

\[
= \left(1 - \frac{1}{x}\right)^2 \left(\frac{x^2 - 2x + 1 - 1}{x^2 (1 - \frac{1}{x})^2}\right)
\]

\[
= \left(1 - \frac{1}{x}\right)^2 \left(\frac{x^2 - 2x}{x^2 (1 - \frac{1}{x})^2}\right)
\]

\[
= \frac{x - 2}{x}
\]

\[
= 1 - \frac{2}{x}
\]

\[
= \text{LHS}.
\]

\[\square\]

**Corollary 1.1.** For \(\text{Re} \, s > 1\)

\[
\sum_{n=1}^{\infty} \frac{\mu(n) d(n)}{n^s} = \prod_p \left(1 - \frac{2}{p^s}\right) = \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right)^2 \prod_p \left(1 - \frac{1}{p^{2s} (1 - \frac{1}{p^s})^2}\right)
\]

where the last product converges uniformly for \(\text{Re} \, s > \frac{1}{2}\).

**Proof.** We use the product formula ([1], p 231) for \(\text{Re} \, s > 1\)

\[
\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)
\]

and the convergence condition for infinite products:

\[
\prod_n (1 - \alpha_n), \ 0 \leq \alpha_n < 1 \text{ converges iff } \sum_n \alpha_n \text{ converges.} \quad \square
\]

**Remark 1.1.** Recall([1],p 97) that the Prime Number Theorem is equivalent to

\[
\sum_n \frac{\mu(n)}{n} = 0
\]

**Lemma 1.3.** The Cauchy product

\[
\sum_{n=1}^{\infty} \frac{a_n}{n} = \left(\sum_{l=1}^{\infty} \frac{\mu(l)}{l}\right) \left(\sum_{m=1}^{\infty} \frac{\mu(m)}{m}\right)
\]

converges to 0.
\textbf{Proof.} We note that the formal Cauchy product holds at $s = 1$ and that $a_n = \sum_{lm=n} \mu(l)\mu(m)$ as in ([3], 4.4.5). We use the estimate there:

(i) $\sum_{n \leq x} a_n = t_x = \mathcal{O}(x \exp(-c(\log x)\frac{1}{10}))$ so that $t_x = o(x)$ and $|t_x| \leq Ke^{-c(\log x)\frac{1}{10}}$ for suitable positive constants $c, K$.

(ii) By theorem 9.63(c), $\sum \frac{a_n}{n}$ is Cesaro summable if and only if $\sum \frac{t_n}{n(n+1)}$ converges. We check absolute convergence by the integral test, applying the above estimate.

$$\int_1^{\infty} \frac{t_x}{x(x+1)} dx \leq K \int_1^{\infty} \frac{1}{(x+1)} \frac{1}{e^{c(\log x)\frac{1}{10}}} dx \leq K \int_1^{\infty} \frac{1}{x \ e^{c(\log x)\frac{1}{10}}} dx$$
changing variables to $y = \log x$ we have the convergence of the integral.

(iii) Now "$t_n = o(n)$" and Cesaro summability together imply the convergence of the series $\sum \frac{a_n}{n}$ (Theorem 9.63(b), [2]). The value of the sum is 0 since all three series above are convergent and $\sum \frac{\mu_n}{n} = 0$ (Prime Number Theorem) so that the sum on the left is $(0)(0) = 0$.

\textbf{Remark 1.2.} We may derive our main claim "$\sum \frac{\mu(n)d(n)}{n} = 0$" by the same argument as in Lemma 1.5 using [3], 4.4.6, in particular using $\sum_{n \leq x} \mu(n)d(n) = o(x)$. But our way shows a link with the Prime Number Theorem. We show that our convergence statement is equivalent to the Prime Number Theorem(Prop 1.11 below). $\square$

\textbf{Proposition 1.1.}

$$\sum_{n} \frac{\mu(n)d(n)}{n} = 0$$

\textbf{Proof.} By Cor 1.3 with $s = 1$ we have the formal Cauchy product equality since the partial sums are limits as $s \to 1^+$

$$\sum_{n} \frac{\mu(n)d(n)}{n} = \left(\sum_{n} \frac{\mu(n)}{n}\right)^2 \left(\sum_{n} \frac{b_n}{n}\right)$$

$\sum \frac{b_n}{n}$ converges absolutely to a positive limit $L$ and $\left(\sum \frac{\mu(n)}{n}\right)^2 = 0$ by Lemma 1.5. So by Mertens’ Criterion(if one of the series is absolutely convergent and the other is convergent then the Cauchy product converges to the product of the limits) the Cauchy product converges to $(0)(L) = 0$ as claimed. $\square$

\textbf{Corollary 1.2.}

$$\sum_{n \leq x} \mu(n)d(n) = o(x)$$

\textbf{Proof.} We apply([2], Theorem 9.6.3) let $s_n = \sum \frac{n}{i=1} u_i, \sigma_n = \frac{1}{n} \sum_{i=1}^{n} s_i$
\[ t_n = u_1 + 2u_2 + \cdots + nu_n \]
\[ = (n + 1)s_n - n\sigma_n \]

Hence
\[ \lim_{n \to \infty} \frac{t_n}{n} = \lim_{n \to \infty} \frac{(n + 1)}{n} s_n - \lim_{n \to \infty} \sigma_n \]
\[ = (1)(L) - L = 0 \]

Take \( u_n = \frac{\mu(n)d(n)}{n} \) to obtain \( \sum_{n \leq x} \mu(n)d(n) = o(x) \).

**Lemma 1.4.** \( \sum_{n} \frac{\mu(n)d(n)}{n} = 0 \) implies \( \sum_{n} \frac{a_n}{n} = (\sum \frac{\mu(m)}{m})(\sum \frac{\mu(l)}{l}) \) converges to 0.

**Proof.** As above we use Mertens on
\[ \left( \sum \frac{\mu(n)d(n)}{n} \right) \left( \sum \frac{c_n}{n} \right) = \sum \frac{a_n}{n} \]
where \( \sum c_n = \prod_p \left( 1 - \frac{1}{p^{2\mu(1)/p^2}} \right)^{-1} \), inverse of the earlier product.

**Lemma 1.5.** \( \sum a_n/n = 0 \) implies \( \sum \frac{\mu(n)}{n} = 0 \) (the Prime Number Theorem).

**Proof.** Consider the Power series \( \sum \frac{a_n x^n}{n} \) and apply Abel’s Convergence Theorem to obtain uniform convergence on an interval \([x_0, 1]\). Write the power series as the sum of partial sum and remainder
\[ \sum \frac{a_n}{n} x^n = s_k(x) + R_k(x) \]

with
\[ s_k(x) = \sum_{i+j \leq k} p_i(x)p_j(x) \]
\[ R_k(x) = \sum_{i+j > k} p_i(x)p_j(x) \]
\[ p_m(x) = \sum_{n=1}^m \frac{\mu(n)}{n} x^n = \text{partial sum of } \sum \frac{\mu(n)}{n} x^n \]

Now uniform convergence on \([x_0, 1]\) implies \( R_k(x) \to 0 \) uniformly as \( k \to \infty \). Hence the "\( n^{th} \) term" in the convergent series for \( R_k(x) \) tends to 0 uniformly. Choosing \( x = 1 \) and the subsequence with \( i = j \), we have \( p_i^2(1) \to 0 \) and \( p_i(1) \to 0 \) as \( i \to \infty \) which is the Prime Number Theorem(The use of power series may be avoided by considering \( x = 1 \) and only \( s_k(1) + R_k(1) \) etc).

**Proposition 1.2.** \( \sum_{n} \frac{\mu(n)d(n)}{n} = 0 \) is equivalent to the Prime Number Theorem.

**Proof.** Combine Remark 1.4, Prop 1.7 and Lemmas 1.9, 1.10.
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References

SURVEY ON THE KAKUTANI PROBLEM IN P-ADIC ANALYSIS I

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ABSTRACT. Let $\mathbb{K}$ be a complete ultrametric algebraically closed field and let $\mathcal{A}$ be the Banach $\mathbb{K}$-algebra of bounded analytic functions in the "open" unit disk $D$ of $\mathbb{K}$ provided with the Gauss norm. Let $\text{Mult}(\mathcal{A}, \| \cdot \|)$ be the set of continuous multiplicative semi-norms of $\mathcal{A}$ provided with the topology of pointwise convergence, let $\text{Mult}_m(\mathcal{A}, \| \cdot \|)$ be the subset of the $\phi \in \text{Mult}(\mathcal{A}, \| \cdot \|)$ whose kernel is a maximal ideal and let $\text{Mult}_1(\mathcal{A}, \| \cdot \|)$ be the subset of the $\phi \in \text{Mult}(\mathcal{A}, \| \cdot \|)$ whose kernel is a maximal ideal of the form $(x-a)\mathcal{A}$ with $a \in D$. By analogy with the Archimedean context, one usually calls ultrametric Corona problem, or ultrametric Kakutani problem the question whether $\text{Mult}_1(\mathcal{A}, \| \cdot \|)$ is dense in $\text{Mult}_m(\mathcal{A}, \| \cdot \|)$. In order to recall the study of this problem that was made in several successive steps, here we first recall how to characterize the various continuous multiplicative semi-norms of $\mathcal{A}$, with particularly the nice construction of certain multiplicative semi-norms of $\mathcal{A}$ whose kernel is neither a null ideal nor a maximal ideal, due to J. Araujo. Here we prove that multbijecitivity implies density. The problem of multbijecitivity will be described in a further paper.

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1. INTRODUCTION AND RESULTS.

Let $T = H^\infty(B)$ be the unital Banach algebra of bounded analytic functions on the open unit disk $B$ in the complex plane. Each $a \in B$ defines a multiplicative linear functional $\phi_a$ on $T$ by "point evaluation" i.e. $\phi_a(f) = f(a)$. If a function $f$ lies in the kernel of all the $\phi_a$ then clearly $f = 0$. This tells us that the set of all the $\phi_a$ is dense in the set $\Xi(T)$ of all non-zero multiplicative linear functionals on $T$ in the hull-kernel topology which is lifted from the kernels of the functionals, which are the maximal ideals of $T$ (each maximal ideal, being of codimension 1, is the kernel of a multiplicative linear functional).

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The Corona Conjecture of Kakutani was that one also has density with respect to the weak topology (or Gelfand topology) which is the topology of pointwise convergence on $T$, defined on the space $Ξ(T)$. This was famously proved by Carleson in 1962 [4]. The key fact is that if $f_1, \ldots, f_n$ belong to $T$ and if there exists $d > 0$ such that, for all $a \in B$ we have

$$|f_1(z)| + \ldots + |f_n(z)| > d$$

then the ideal generated by the $f_1, \ldots, f_n$ is the whole of $T$. People often transfer the name "Corona Statement" to this key fact. Indeed, this Corona Statement implies that the Corona Conjecture is true, thanks to the fact that all maximal ideals of a $C$-Banach algebra are of codimension 1.

Now consider the situation in the non-archimedean context.

**Notations.** Let $IK$ be an algebraically closed field complete with respect to an ultrametric absolute value $| \cdot |$. Given $a \in IK$ and $r > 0$, we denote by $d(a, r)$ the disk $\{x \in IK \mid |x - a| \leq r\}$, by $d(a, r^-)$ the disk $\{x \in K \mid |x - a| < r\}$, by $C(a, r)$ the circle $\{x \in IK \mid |x - a| = r\}$ and set $D = d(0, 1^-)$.

Let $a \in D$. Given $r, s \in [0, 1]$ such that $0 < r < s$ we set $\Gamma(a, r, s) = \{x \in IK \mid |x - a| \leq r < s\}$.

Let $A$ be the $IK$-algebra of bounded power series converging in $D$ which is complete with respect to the Gauss norm defined as $\| \sum_{n=1}^{\infty} a_n x^n \| = \sup_{n \in IN} |a_n|$: we know that this norm actually is the norm of uniform convergence on $D$ [5].

In [16] the Corona problem was considered in a similar way as it is on the field $C$ [4]: the author asked the question whether the set of maximal ideals of $A$ defined by the points of $D$ (which are well known to be of the form $(x - a)A$), is dense in the whole set of maximal ideals with respect to a so-called "Gelfand Topology". In fact, as explained in [8], this makes no sense because the maximal ideals which are not of the form $(x - a)A$ are of infinite codimension [8]. Consequently, a Corona problem should be defined in a different way, as explained in [8]. Actually, one can’t define a relevant topology on the maximal spectrum of a Banach $IK$-algebra having maximal ideals of infinite codimension: the only spectrum we have to consider is Guennebaud’s spectrum of Continuous multiplicative semi-norms [13], which is at the basis of Berkovich theory [2].

However, in [16] a "Corona Statement" similar to that mentioned above was shown in our algebra $A$ and it is useful in the present paper as it was in [8]. Roughly, the "Corona Statement" shows that each maximal ideal is just the ideal of elements of the algebra $T$, vanishing along an ultrafilter, on the domain $D$. Therefore, on $C$, $f(z)$ has a limit along the ultrafilter and the limit defines a character which, by definition, lies in the closure of the set of characters defined by points of $D$. And there are no other characters. On the field $IK$, although a similar "Corona Statement" remains true [16], we can’t manage the problem in the same way because
$f(x)$ has no limit along an ultrafilter (the field is not locally compact). But we may consider continuous multiplicative semi-norms and then $|f(x)|$ has a limit along an ultrafilter, which defines again a continuous multiplicative semi-norm. The role of ultrafilters then appears essential to define the multiplicative semi-norms of $A$. But do we get all continuous multiplicative semi-norms whose kernel is a maximal ideal, in that way? That is the problem (easily solved when the field is strongly valued [9] and next, solved when IK is spherically complete [9]). Here we will recall that if $A$ is multbijective, then each continuous multiplicative semi-norms whose kernel is a maximal ideal lies in the closure of the set of multiplicative semi-norms defined by points of $D$. The hard problem of multbijectivity will be examined in a further article.

On the other hand, we will show that certain continuous multiplicative semi-norms have a kernel that is neither null nor a maximal ideal: followin the method due to J. Araujo, we call them Araujo’s semi-norms [1].

Recall classical results on analytic elements. Let $E$ be a closed bounded subset of IK. We denote by $R(E)$ the algebra of rational functions having no pole in $E$ and we denote by $H(E)$ the Banach IK-algebra completion of $R(E)$ with respect to the norm of uniform convergence $\| \cdot \|_E$ which is defined on $R(E)$ because every rational function is bounded on such a set $E$.

Theorem I.1 summerizes a few classical properties of analytic elements in a disk $d(a,R)$ [5].

**Theorem I.1.** Let $a \in IK$ and let $r > 0$. Then $H(d(a,r))$ is the set of power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ such that $\lim_{n \to +\infty} |a_n|r^n = 0$. Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n \in H(d(a,R))$ and let $q$ be the biggest integer such that $|a_q|r^q = \max_{n \in \mathbb{N}} |a_n|r^n$. Then the number of zeros of $f$ in $d(a,r)$ is equal to $q$.

Given a maximal ideal of $H(d(a,R))$, it is of the form $(x-b)H(d(a,R))$.

We have defined by $A$ the IK-algebra of power series $f = \sum_{n=0}^{\infty} a_nx^n$ such that $\sup_{n \in \mathbb{N}} |a_n| < +\infty$. Then each element of $A$ converges in the disk $D$. Now, fixing $a \in D$, then $f$ is also equal to a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ which also converges in $D$. We denote by $\| \cdot \|$ the norm of uniform convergence on $D$.

**Notations.** Let $S$ be a unital commutative normed algebra whose norm is denoted by $\| \cdot \|$. We denote by $Mult(S,\| \cdot \|)$ the set of continuous multiplicative semi-norms of $S$. For each $\phi \in Mult(S,\| \cdot \|)$, we denote by $\text{Ker}(\phi)$ the closed prime ideal of the $f \in S$ such that $\phi(f) = 0$. The set of the $\phi \in Mult(S,\| \cdot \|)$ such that $\text{Ker}(\phi)$ is a maximal ideal is denoted by $Mult_m(S,\| \cdot \|)$, the set of the $\phi \in Mult(T,\| \cdot \|)$ such that $\text{Ker}(\phi)$ is a maximal ideal of codimension 1 is denoted by $Mult_1(S,\| \cdot \|)$ [2], [13].
First, we will characterize all continuous multiplicative norms on \( A \). Next, recalling Araujo’s construction, we will present continuous multiplicative semi-norms whose kernel is a prime closed ideal that is neither null nor maximal [1].

The ultrametric Corona problem may be viewed at two levels:
1) Is \( \text{Mult}_1(A, \| \cdot \|) \) dense in \( \text{Mult}_m(A, \| \cdot \|) \) (with respect to the topology of pointwise convergence)?
2) Is \( \text{Mult}_1(A, \| \cdot \|) \) dense in \( \text{Mult}(A, \| \cdot \|) \) (with respect to the same topology)?

In a further article, we will try to solve Question 1). Actually, this way to set the Corona problem on an ultrametric field is not really different from the original problem once considered on \( \mathbb{C} \) because on a commutative unital Banach \( \mathbb{C} \)-algebra \( S \), all continuous multiplicative semi-norms are known to be of the form \( |\chi| \) where \( \chi \) is a character of \( S \). Thus the Corona problem was equivalent to show that the set of multiplicative semi-norms defined by the points of the open disk of center 0 and radius 1 was dense inside the whole set of continuous multiplicative semi-norms, with respect to the topology of pointwise convergence.

Let us recall some classical results on multiplicative semi-norms in ultrametric Banach algebras. Let \( B \) be a commutative unital Banach \( \mathbb{K} \)-algebra. We know that for every \( \mathcal{M} \in \text{Max}(B) \), there exists at least one \( \phi \in \text{Mult}_m(B, \| \cdot \|) \) such that \( \text{Ker}(\phi) = \mathcal{M} \) but in certain cases, there exist infinitely many \( \phi \in \text{Mult}_m(B, \| \cdot \|) \) such that \( \text{Ker}(\phi) = \mathcal{M} \), [6], [7].

A maximal ideal \( \mathcal{M} \) of \( B \) is said to be univalent if there is only one \( \phi \in \text{Mult}_m(B, \| \cdot \|) \) such that \( \text{Ker}(\phi) = \mathcal{M} \) and the algebra \( B \) is said to be multbijective if every maximal ideal is univalent. It was proven that non-multbijective commutative unital Banach \( \mathbb{K} \)-algebras with unity do exist [6], [7]. The question whether \( A \) is multbijective here appears to be crucial.

**Remark.** Given a filter \( G \), if for every \( f \in A \), \( |f(x)| \) admits a limit \( \varphi_G(f) \) along \( G \), the function \( \varphi_G \) obviously belongs to \( \text{Mult}(A, \| \cdot \|) \). Moreover, it clearly lies in the closure of \( \text{Mult}_1(A, \| \cdot \|) \). Consequently, if we can prove that every element of \( \text{Mult}_m(A, \| \cdot \|) \) is of the form \( \varphi_G \), with \( G \) a certain filter on \( D \), Question 1) is solved.

Thus it is important to know the nature of continuous multiplicative semi-norms on \( A \). Unfortunately, we can’t give a complete characterization.

In the proof of Theorems we shall need several basic results. Lemma I.2 is immediate and Lemma I.3 is well known [9]:

**Lemma I.2.** Let \( \sum_{n=0}^{\infty} u_n \) be a converging series with positive terms. There exists a sequence of strictly positive integers \( t_n \in \mathbb{N} \) satisfying
\[
 t_n \leq t_{n+1}, \quad n \in \mathbb{N}, \\
 \lim_{n \to \infty} t_n = +\infty, \\
 \sum_{m=0}^{\infty} t_m u_m < +\infty.
\]
Definition. An element \( f \in A \) will be said to be quasi-invertible if it factorizes in \( A \) in the form \( P(x)g(x) \) where \( P \) is a polynomial whose zeros lie in \( D \) and \( g \) is a quasi-invertible element of \( A \).

Lemma I.3. Let \( f \in A \) be not quasi-invertible and let \( (a_n)_{n \in \mathbb{N}} \) be the sequence of its zeros with respective multiplicity \( q_n \). Then the series \( \sum_{n=0}^{\infty} q_n \log(|a_n|) \) converges to \( \log(|f(0)|) - \log \|f\| \).

Now, we have the following Theorem (Theorem 14.6 and Corollary 14.10 in [10]):

Theorem I.4. Given \( f = \sum_{n=0}^{\infty} a_n x^n \in A \), we have \( \|f\| = \sup_{n \in \mathbb{N}} |a_n| = \sup \{ \phi(f) \mid \phi \in \text{Mult}(A, \|\cdot\|) \} \). The norm \( \|f\| \) is multiplicative and every \( f \in A \) is uniformly continuous in \( D \). For every \( r \in ]0,1[ \), \( f \) has finitely many zeros in \( d(0,r^-) \). Let \( a \in C(0,r) \). If \( f \) has no zero in \( d(a,r^-) \) then \( |f(x)| = |f(r)| \) \( \forall x \in d(a,r^-) \).

Moreover, the three following statements are equivalent,
1) \( f \) has no zero in \( D \),
2) \( f \) is invertible in \( A \),
3) \( \|f\| = |a_0| \).

\( |f(x)| \) is a constant in \( D \).

Corollary I.4.1. An element \( f \in A \) is quasi-invertible if and only if it has finitely many zeros.

Lemmas I.5, I.6 are also classical and particularly, are given in [5] and particularly in Theorem 28.1 of [10].

Lemma I.5. Let \( f, g \in A \) be such that every zero \( a \) of \( f \) is a zero of \( g \) of order superior or equal to its order as a zero of \( f \). Then there exists \( h \in A \) such that \( g = fh \).

Lemma I.6. Let \( f \in A \). Then \( |f'|(r) \leq \frac{|f(r)|}{r} \) \( \forall r < 1 \).

By classical results on analytic functions we know this lemma (for instance Lemma I.7 is Theorem 22.26 in [10]):

Lemma I.7. Let \( r', r'' \in ]0,1[ \) and let \( f \in A \) admit zeros \( a_1, \ldots, a_q \) of respective order \( k_j, j = 1, \ldots, q \) in \( \Gamma(0,r',r'') \) and no zero in \( d(0,r') \). Then
\[
|f|(r'') = |f|(r') \prod_{j=1}^{q} \left( \frac{r''}{|a_j|} \right)^{k_j}.
\]

Lemma I.8. Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \in H(C(0,r)) \) and assume that \( f \) has a unique zero \( \alpha \), of order 1, in \( C(0,r) \). Then \( |f'((\alpha))| = |f|(r) \).

Proof. By hypothesis, \( f(x) \) is of the form \( (x - \alpha)h(x) \) with \( h \in H(C(0,r)) \), having no zero in \( C(0,r) \). Then \( |f|(r) = r|h|(r) \). Moreover, since \( h \) has no zero in \( C(0,r) \), we have \( |h(\alpha)| = |h|(r) \). And by Lemma I.6, \( |f'|(r) \leq \frac{|f|(r)}{r} \). Therefore, we have \( |f'((\alpha))| \leq |f'|'(r) \leq \frac{|f|(r)}{r} = |h|(r) = |h(\alpha)| = |f'((\alpha))| \) and hence \( |f'((\alpha))| = |f|(r) \).
Definitions and notation. We call circular filter of center \( a \) and diameter \( R \) on \( D \) the filter \( \mathcal{F} \) which admits as a generating system the family of sets \( \Gamma(\alpha, r', r'') \cap D \) with \( \alpha \in d(a, R), r' < R < r'', \) i.e. \( \mathcal{F} \) is the filter which admits for basis the family of sets of the form \( D \cap \left( \bigcap_{i=1}^{q} \Gamma(\alpha_i, r'_i, r''_i) \right) \) with \( \alpha_i \in d(a, R), r'_i < R < r''_i \) \((1 \leq i \leq q), q \in \mathbb{N})\).

Recall that the field \( \mathbb{K} \) is said to be spherically complete if every decreasing sequence of disks has a non-empty intersection. Each field such as \( \mathbb{K} \) admits an algebraically closed spherical completion (see Theorems 7.4 and 7.6 in [10]).

In a field which is not spherically complete, one has to consider decreasing sequences of disks \( (D_n) \) with an empty intersection. We call circular filter with no center, of canonical basis \( (D_n) \) the filter admitting for basis the sequence \( (D_n) \) and the number \( \lim_{n \to \infty} \text{diam}(D_n) \) is called diameter of the filter.

Finally the filter of neighborhoods of a point \( a \in D \) is called circular filter of the neighborhoods of \( a \) on \( D \) and its diameter is 0. Given a circular filter \( \mathcal{F} \), its diameter is denoted by \( \text{diam}(\mathcal{F}) \).

Given \( a \in \mathbb{K} \) and \( r > 0 \), we denote by \( \Phi(a, r) \) the set of circular filters secant with \( d(a, r) \) i.e. the circular filters of center \( b \in d(a, r) \) and radius \( s \in [0, r] \).

Here, we will denote by \( \mathcal{W} \) the circular filter on \( D \) of center 0 and diameter 1 and by \( \mathcal{Y} \) the filter admitting for basis the family of sets of the form \( \Gamma(0, r, 1) \setminus \left( \bigcup_{n=0}^{\infty} d(a_n, r_n) \right) \) with \( a_n \in D, r_n \leq |a_n| \) and \( \lim_{n \to \infty} |a_n| = 1 \).

On \( \mathbb{K}[x] \), circular filters on \( \mathbb{K} \) are known to characterize multiplicative semi-norms by associating to each circular filter \( \mathcal{F} \) the multiplicative semi-norm \( \varphi_{\mathcal{F}} \) defined as \( \varphi_{\mathcal{F}}(f) = \lim_{r \to 0} |f(x)| \) \([12], [13], [9], [11]\).

We know that every \( f \in A \) is an analytic element in each disk \( d(a, r) \) whenever \( r \in [0, 1] \) \([5]\). Consequently, by classical results \([5]\), several properties of polynomials have continuation to analytic elements and to \( A \).

Definitions and notation. Let \( a \in D \) and let \( R \in [0, 1] \). Given \( r, s \in \mathbb{R} \) such that \( 0 < r < s \) we set \( \Gamma(a, r, s) = \{ x \in \mathbb{K} | r < |x - a| < s \} \).

We call circular filter of center \( a \) and diameter \( R \) on \( D \) the filter \( \mathcal{F} \) which admits as a generating system the family of sets \( \Gamma(\alpha, r', r'') \cap D \) with \( \alpha \in d(a, R), r' < R < r'' \), i.e. \( \mathcal{F} \) is the filter which admits for basis the family of sets of the form \( D \cap \left( \bigcap_{i=1}^{q} \Gamma(\alpha_i, r'_i, r''_i) \right) \) with \( \alpha_i \in d(a, R), r'_i < R < r''_i \) \((1 \leq i \leq q), q \in \mathbb{N})\).

Recall that the field \( \mathbb{K} \) is said to be spherically complete if every decreasing sequence of disks has a non-empty intersection. Each field such as \( \mathbb{K} \) admits an algebraically closed spherical completion.

In a field which is not spherically complete, one has to consider decreasing sequences of disks \( (D_n) \) with an empty intersection. We call circular filter with no center, of canonical basis \( (D_n) \) the filter admitting for basis the sequence \( (D_n) \) and the number \( \lim_{n \to \infty} \text{diam}(D_n) \) is called diameter of the filter.
Finally the filter of neighborhoods of a point $a \in D$ is called circular filter of the neighborhoods of $a$ on $D$ and its diameter is 0. Given a circular filter $\mathcal{F}$, its diameter is denoted by $\text{diam}(\mathcal{F})$.

Given $a \in \mathbb{K}$ and $r > 0$, we denote by $\Phi(a,r)$ the set of circular filters secant with $d(a,r)$ i.e. the circular filters of center $b \in d(a,r)$ and radius $s \in [0,r]$.

On $\mathbb{K}[x]$, circular filters on $\mathbb{K}$ are known to characterize multiplicative semi-norms by associating to each circular filter $\mathcal{F}$ the multiplicative semi-norm $\varphi_{\mathcal{F}}$ defined as $\varphi_{\mathcal{F}}(f) = \lim_{\mathcal{F}} |f(x)|$ [12], [5], [6].

We know that every $f \in A$ is an analytic element in each disk $d(a,r)$ whenever $r \in [0,1]$ [5]. Consequently, by classical results [5], several properties of polynomials have continuation to analytic elements and to $A$.

Thus, by results on analytic elements, we have Theorem I.9 [5], and Theorem 13.1 in [10]:

**Theorem I.9.** Let $a \in \mathbb{K}$ and $R \in ]0,1[$ and let $r \in [0,R]$. For each circular filter $\mathcal{F} \in \Phi(a,r)$, for each element $f$ of $H(d(a,r))$ (resp. $f \in H(d(a, r^+) ))$, $|f(x)|$ has a limit $\varphi_{a,r}(f) = \varphi_{\mathcal{F}}(f)$ along $\mathcal{F}$. Moreover, the mapping $\varphi_{\mathcal{F}}$ defined on $H(d(a,R))$ (resp. on $H(d(a, r^+))$) is a multiplicative semi-norm continuous with respect to the norm $\| \cdot \|_{d(a,r)}$ (resp. $\| \cdot \|_{d(a, r^+)}$) and is a norm if and only if $r > 0$.

Next, if $b \in d(a,r)$ (resp. $f \in d(a, r^+)$), then $\varphi_{a,r}(f) = \varphi_{b,r}(f)$. Further, the mapping associating to each circular filter $\mathcal{F} \in \Phi(a,r)$ secant with $d(a,R)$ the continuous multiplicative semi-norm $\varphi_{\mathcal{F}}$ is a bijection from $\Phi(a,r)$ onto $\text{Mult}(H(d(a,r)), \| \cdot \|_{d(a,R)})$.

Now, we will denote by $\mathcal{W}$ the circular filter on $D$ of center 0 and diameter 1 and by $\mathcal{Y}$ the filter admitting for basis the family of sets of the form $\Gamma(0,r,1) \setminus (\bigcup_{n=0}^{\infty} d(a_n, r_n^-))$ with $a_n \in D$, $r_n \leq |a_n|$ and $\lim_{n \to \infty} |a_n| = 1$.

Next, $\varphi_{\mathcal{W}}$ defines the Gauss norm on $\mathbb{K}[x]$ because, given a polynomial $P(x) = \sum_{j=0}^{\infty} a_j x^j$, we have $\varphi_{\mathcal{W}}(P) = \max_{0 \leq j \leq q} |a_j|$. Therefore $\varphi_{\mathcal{W}}$ admits a natural continuation to $A$ as $\| \sum_{n=1}^{\infty} a_n x^n \| = \sup_{n \in \mathbb{N}} |a_n|$. However, by [6] we know that this continuation is far from unique.

So, the problem is first to determine whether a multiplicative semi-norms defined on $\mathbb{K}[x]$ by circular filters on $D$, other than the Gauss norm, have a unique continuation to $A$.

Consequently, given a circular filter $\mathcal{F}$ on $D$ of diameter $< 1$, according to Theorem I.9, for every $f \in A$, $|f(x)|$ has a limit along $\mathcal{F}$ denoted by $\varphi_{\mathcal{F}}(f)$ and then $\varphi_{\mathcal{F}}$ is a continuous multiplicative semi-norm on $A$. In particular, given $a \in D$ and $r \in [0, 1[$, if we consider the circular filter $\mathcal{F}$ of center $a$ and diameter $r$, we denote by $\varphi_{a,r}$ the multiplicative semi-norm $\varphi_{\mathcal{F}}$ which actually is defined by $\varphi_{a,r}(f) = \lim_{|x-a| \to r} |f(x)|$ and is a norm whenever $\text{diam}(\mathcal{F}) > 0$. For convenience, if $\mathcal{F}$ is the circular filter of center 0 and diameter $r$, we set $|f|(r) = \varphi_{\mathcal{F}}(f)$. 
Definitions and notations. Let $E$ be a subset of $\mathbb{K}$. If $E$ is bounded, of diameter $r$, we denote by $\overline{E}$ the disk $d(a,r)$, $a \in E$.

For each $a \in E$, we denote by $I_a$ the mapping from $E$ to $\mathbb{R}_+$ defined as $I_a(x) = |x - a|$.

A subset $E$ of $\mathbb{K}$ is said to be infraconnected if for every $a \in E$, the closure in $\mathbb{R}$ of $I_a(E)$ is an interval.

By classical results ([5], Lemma 2.1) we have the following description:

**Lemma I.10.** Let $E$ be a closed bounded subset of $\mathbb{K}$, of diameter $R$. Then $\overline{E} \setminus E$ admits a unique partition by a family of maximal disks $(d(b_j,r_j^-))_{j \in I}$.

Definitions and notations. Let $E$ be a closed bounded subset of $\mathbb{K}$ and let $(d(b_j,r_j^-))_{j \in I}$ be the partition of $\overline{E} \setminus E$ shown in Lemma I.10. The disks $d(b_j,r_j^-))$, $j \in I$ are called the holes of $E$.

We can now recall the famous Mittag-Leffler Theorem for analytic elements due to Marc Krasner [14] and [5], Theorem 15.1:

**Theorem I.11.** (M. Krasner) Let $E$ be a closed and bounded infraconnected subset of $\mathbb{K}$ and let $f \in H(E)$. There exists a unique sequence of holes $(T_n)_{n \in \mathbb{N}}$ of $E$ and a unique sequence $(f_n)_{n \in \mathbb{N}}$ in $H(E)$ such that $f_0 \in H(\overline{E})$, $f_n \in H_0(\mathbb{K} \setminus T_n)$ $(n > 0)$, $\lim_{n \to \infty} f_n = 0$ satisfying

\[
(1) \quad f = \sum_{n=0}^{\infty} f_n \quad \text{and} \quad \|f\|_D = \sup_{n \in \mathbb{N}} \|f_n\|_E.
\]

Moreover for every hole $T_n = d(a_n, r_n^-)$, we have

\[
(2) \quad \|f_n\|_E = \|f_n\|_{\mathbb{K} \setminus T_n} = \varphi_{a_n,r_n}(f_n) \leq \varphi_{a_n,r_n}(f) \leq \|f\|_E.
\]

If $\overline{E} = d(a,r)$ we have

\[
(3) \quad \|f_0\|_E = \|f_0\|_{\overline{E}} = \varphi_{a,r}(f_0) \leq \varphi_{a,r}(f) \leq \|f\|_E.
\]

Let $F = \overline{E} \setminus \bigcup_{n=1}^{\infty} T_n$. Then $f$ belongs to $H(F)$ and its decomposition in $H(F)$ is given again by (1) and then $f$ satisfies $\|f\|_F = \|f\|_E$.

Let us recall the following results [7], [13]:

**Theorem I.12.** Let $B$ be a unital commutative ultrametric Banach $\mathbb{K}$-algebra. Then $\sup\{\phi(f) \mid \phi \in \text{Mult}(B,\| \cdot \|)\} = \lim_{n \to \infty}(\|f^n\|)^{\frac{1}{n}} \forall f \in B$. On the other hand, $\text{Mult}(B,\| \cdot \|)$ is provided with the topology of pointwise convergence and is compact for this topology.

Let us now recall some general results on maximal ideals [7] (Theorems 15.6 and 27.3):

**Theorem I.13.** Let $B$ be a unital cmmutative Banach $\mathbb{K}$-algebra. Each maximal ideal of $B$ is the kernel of at least one continuous multiplicative semi-norm. If $\mathbb{K}$ has a non-countable residue class field or a non-countable value group then each maximal ideal of $B$ is the kernel of a unique continuous multiplicative semi-norm.
Corollary I.13.1. If $\mathbb{K}$ has a non-countable residue class field or a non-countable value group then every unital commutative Banach $\mathbb{K}$-algebra is multibijective.

Definition. The field $\mathbb{K}$ is said to be strongly valued if one of the following two sets are not countable:
1) the residue field of $\mathbb{K}$,
2) the set $|\mathbb{K}| = \{|x| \mid x \in \mathbb{K}\}$.

Remark. When the algebraically closed complete field is not strongly valued, there exist Banach $\mathbb{K}$-algebras which are not multibijective [6], [7].

2. Multiplicative spectrum and maximal ideals

Let us first notice the following basic result which is indispensable in the sequel:

Theorem II.1. Let $F$ be a filter on $D$ such that for all $f \in A$, $|f(x)|$ have a limit along $F$. Then the mapping $\phi$ defined on $A$ as $\phi(f) = \lim_F |f(x)|$ belongs to the closure of $\text{Mult}_1(A, \| \cdot \|)$ in $\text{Mult}(A, \| \cdot \|)$.

Proof. Let us first notice that given an element $\phi \in \text{Mult}_1(A, \| \cdot \|)$ of the form $\varphi_F$, with $F$ a filter on $D$, then $\phi$ clearly belongs to the closure of $\text{Mult}_1(A, \| \cdot \|)$ in $\text{Mult}(A, \| \cdot \|)$. More precisely, take $f_1, \ldots, f_q \in A$ and $\varepsilon > 0$. For each $j = 1, \ldots, q$ there exists $B_j \in F$ such that $|f_j(x)| - |\varphi_{B_j}(f_j)| \leq \varepsilon \forall x \in B_j$. Let $B = \bigcap_{j=1}^q B_j$. Then $|f_j(x)| - |\varphi_{B_j}(f_j)| \leq \varepsilon \forall x \in B$, $\forall j = 1, \ldots, q$, which ends the proof. \hfill $\Box$

We will now apply to the algebra $A$ all results already known concerning algebras $H(d(a,R))$ and $H(d(a,R^-))$.

Now, when studying the set of multiplicative semi-norms of the algebra $A$, we have to consider coroner ultrafilters.

Definitions and notations. An ultrafilter $\mathcal{U}$ on $D$ will be called coroner ultrafilter if it is thinner than $\mathcal{W}$. Similarly, a sequence $(a_n)$ on $D$ will be called a coroner sequence if its filter is a coroner filter, i.e. if $\lim_{n \to +\infty} |a_n| = 1$.

Two coroner ultrafilters $\mathcal{F}$, $\mathcal{G}$ are said to be contiguous if for every subsets $F \in \mathcal{F}$, $G \in \mathcal{G}$ of $D$ the distance from $F$ to $G$ is null.

Let $\psi \in \text{Mult}_1(A, \| \cdot \|)$ be different from $\| \cdot \|$. Then $\psi$ will be said to be coroner if its restriction to $\mathbb{K}[x]$ is equal to $\| \cdot \|$. In [8] regular ultrafilters were defined. Let $(a_n)_{n \in \mathbb{N}}$ be a coroner sequence in $D$. The sequence is called a regular sequence if $\inf_{j \in \mathbb{N}} \prod_{n \in \mathbb{N}, n \neq j} |a_n - a_j| > 0$.

An ultrafilter $\mathcal{U}$ is said to be regular if it is thinner than a regular sequence. Thus, by definition, a regular ultrafilter is a coroner ultrafilter.

Now, given an ultrafilter $\mathcal{U}$ on $D$, the function $|f(x)|$ from $D$ to $[0, \| f \|]$ has a limit $\varphi_{\mathcal{U}}(f)$ which clearly defines an element of $\text{Mult}_1(A, \| \cdot \|)$. We can then derive Theorem II.2.
Given a filter \( \mathcal{F} \) on \( D \), we will denote by \( J(\mathcal{F}) \) the ideal of the \( f \in A \) such that \( \lim_{\mathcal{F}} f(x) = 0 \).

**Theorem II.2.** Let \( \mathcal{U} \) be an ultrafilter on \( D \). For every \( f \in A \), \( |f(x)| \) admits a limit \( \phi(\mathcal{U})(f) \) along \( \mathcal{U} \). Moreover, the mapping \( \phi(\mathcal{U}) \) from \( A \) to \( \mathbb{R}_+ \) belongs to \( \text{Multi}(A, \| . \|) \) and \( \text{Ker}(\phi(\mathcal{U})) = J(\mathcal{U}) \). Given two contiguous ultrafilters \( \mathcal{U}_1, \mathcal{U}_2 \) on \( D \), \( \phi(\mathcal{U}_1) = \phi(\mathcal{U}_2) \).

**Proof.** Let \( \Theta \) be the function defined in \( D \), by \( \Theta(x) = |f(x)| \). For each \( f \in A \), \( \Theta \) takes values in the compact \([0, \|f\|]\). Clearly, \( \Theta \) admits a limit \( \phi(\mathcal{U}) \) along every ultrafilter \( \mathcal{U} \) on \( D \). Consequently, \( \phi(\mathcal{U}) \) defines a continuous multiplicative seminorm on \( A \) whose kernel is \( J(\mathcal{U}) \). Finally, since every function \( f \in A \) is uniformly continuous, it is easily seen that \( \lim_{\mathcal{U}_1} |f(x)| = \lim_{\mathcal{U}_2} |f(x)| \forall f \in A \), hence \( \phi(\mathcal{U}_1)(f) = \phi(\mathcal{U}_2)(f) \forall f \in A \), hence \( \phi(\mathcal{U}_1) = \phi(\mathcal{U}_2) \). \( \square \)

**Remark.** Contrary to the context of uniformly continuous bounded functions [11], it seems very hard to know whether two ultrafilters \( \mathcal{U} \) and \( \mathcal{V} \) on \( D \) such that \( \phi(\mathcal{U}) = \phi(\mathcal{V}) \) are contiguous.

**Notation.** We will denote by \( \mathcal{Y} \) the filter on \( D \) admitting for basis the family of sets of the form

\[
d(0, 1^-) \setminus \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{q_n} d(a_{n,j}, r_n^-)
\]

with \( |a_{j,n}| = r_n < r_{n+1} < \forall n \in \mathbb{N} \) and \( \lim_{n \to \infty} r_n = 1 \).

**Proposition II.3.** For every \( f \in A \), \( \|f\| = \lim_{\mathcal{F}} |f(x)| \).

**Proof.** Let \( f \in A \), let \( (C(0, r_n))_{n \in \mathbb{N}} \) be the sequence of circles of center 0 containing zeros of \( f \) and for every \( n \in \mathbb{N} \), let \( a_{n,1}, \ldots, a_{n,q_n} \) be the zeros of \( f \) in \( C(0, r_n) \). Then we can see that \( |f(x)| = |f(|x|)| \forall x \notin d(0, 1^-) \setminus \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{q_n} d(a_{n,j}, r_n^-) \).

Next, by definition of the norm \( \| . \| \), we have \( \|f\| = \lim_{r \to 1} |f(r)| \), which ends the proof. \( \square \)

**Proposition II.4.** Let \( \phi \in \text{Multi}(A, \| . \|) \) satisfy \( \phi(P) = \|P\| \forall P \in \mathbb{I} \mathbb{K}[x] \). Every quasi-invertible element \( f \in A \) also satisfies \( \phi(f) = \|f\| \).

**Proof.** First suppose \( f \in A \) invertible in \( A \). Then \( 1 = \phi(f)\phi(f^{-1}) \). But \( \phi(f) \leq \|f\|, \phi(f^{-1}) \leq \|f^{-1}\| \), hence both inequalities must be equalities. Now, let \( f = Pg \in A \) be quasi-invertible, with \( P \in \mathbb{I} \mathbb{K}[x] \) a polynomial having all zeros in \( D \) and \( g \in A \), invertible in \( A \). Then \( \phi(f) = \phi(P)\phi(g) = \|P\|\|g\| = \|Pg\| = \|f\| \). \( \square \)

Let us now look at the maximal spectrum of \( A \).

**Theorem II.5.** Let \( \mathcal{M} \) be an ideal of \( A \). Then \( \mathcal{M} \) is a maximal ideal of codimension 1 if and only if it is of the form \( (x - a)A \) with \( a \in D \).

**Proof.** Given \( a \in D \), the ideal \( (x - a)A \) is obviously a maximal ideal of codimension 1 because the mapping \( \chi_a \) from \( A \) to \( \mathbb{I} \mathbb{K} \) defined as \( \chi_a(f) = f(a) \) maps \( A \) onto \( \mathbb{I} \mathbb{K} \).
Now, let \( \mathcal{M} \) be a maximal ideal of codimension 1 and let \( \theta \) be the \( \mathcal{I}K \)-algebra homorphism from \( A \) onto \( \mathcal{I}K \) admitting \( \mathcal{M} \) for kernel. Let \( a = \theta(x) \). Since \( \theta(x - a) = \theta(x) - a \) is not invertible, we have \( |a| < 1 \) because if \( |a| \geq 1 \), then \( \frac{1}{x-a} \) belongs to \( H(D) \) and hence to \( A \). Thus, \( a \) belongs to \( D \). We know that all characters of a Banach \( \mathcal{I}K \)-algebra are continuous (see for instance Theorem 6.19 in [7]), hence so is \( \theta \). Consequently, \( \theta(f) = f(a) \quad \forall f \in A \) and hence \( \text{Ker}(\theta) = (x-a)A \).

**Notation.** We will denote by \( \text{Mult}_1(A, \| \cdot \|) \) the set of maximal ideals of codimension 1.

**Remark.** \( A \) admits maximal ideals of infinite codimension.

**Theorem II.6.** Let \( M \) be a maximal ideal of \( A \). The following statements are equivalent:

(i) there exists \( a \in D \) such that \( M = (x-a)A \),

(ii) \( M \) is principal,

(iii) \( M \) is of finite type,

(iv) \( M \) is of codimension 1.

(v) \( M \) contains a quasi-invertible element.

**Proof.** Suppose (i) is satisfied. Then so are (ii) and (iii) and by Theorem I.4, so is (iv). Moreover, by(i), \( x-a \) belongs to \( A \), hence (v) is satisfied. Suppose now that (v) is satisfied and let \( P(x)g(x) \in M \) be quasi-invertible, with \( P \) a polynomial whose zeros lie in \( D \) and \( g \) an invertible element of \( A \). Then \( P \) belongs to \( M \). Let \( P(x) = \prod_{j=1}^{q}(x-a_j) \). Since \( M \) is prime, one of the \( x-a_j \) belongs to \( D \) and hence (i) is satisfied, which ends the proof.

**Corollary II.6.1.** An element \( \varphi \) of \( \text{Mult}(A, \| \cdot \|) \) belongs to \( \text{Mult}_1(A, \| \cdot \|) \) if and only if there exists \( \alpha \in D \) such that \( \varphi(f) = |f(\alpha)|, \quad \forall f \in A \).

**Notation.** Given an ideal \( I \) of \( A \) we will denote by \( G_I \) the filter generated by the sets \( E(f, \varepsilon), \ f \in I, \varepsilon > 0 \). By definition, \( G_I \) is minimal, with respect to the relation of thinness, among the filters \( \mathcal{H} \) such that \( \lim_{\mathcal{H}} f(x) = 0 \quad \forall f \in I \).

**Theorem II.7.** Let \( M \) be a non-principal maximal ideal of \( A \). Then \( M = J(G_M) \).

**Proof.** By definition, we have \( M \subset J(G_M) \). On the other hand, \( J(G_M) \neq A \) because by Theorem II.5 all elements of \( J(G_M) \) are non-quasi-invertible. Consequently, \( M = J(G_M) \). □

**Corollary II.7.1:** Let \( M \) be a non-principal maximal ideal of \( A \). For every ultrafilter \( \mathcal{U} \) thinner than \( G_M \), \( J(\mathcal{U}) = M \).

**Corollary II.7.2.** For every maximal ideal \( M \) of \( A \), there exist ultrafilters \( \mathcal{U} \) such that \( M = J(\mathcal{U}) \).

**Proof.** Indeed, either \( M \) is principal, of the form \( (x-a)A \) or \( M \) is not principal and then the answer comes from Corollary II.7.1. □
Definition. A maximal ideal \( M \) of \( A \) will be said to be coroner (resp. regular) if there exists a coroner (resp. regular) ultrafilter \( \mathcal{U} \) such that \( M = \mathcal{J}(\mathcal{U}) \).

Notation. Let \( \mathcal{H} \) be the \( \mathbb{K} \)-algebra of (bounded or not) analytic functions in \( D \).

Theorem II.8 also is classical ([10], Theorem 22.26):

Theorem II.8. Let \( f \in \mathcal{H} \) and let \( r_1, r_2 \in ]0,1[ \) satisfy \( r_1 < r_2 \). If \( f \) admits exactly \( q \) zeros in \( d(O, r_1) \) (taking multiplicity into account) and \( t \) different zeros \( \alpha_j \), of respective multiplicity order \( m_j \) \((1 \leq j \leq t)\) in \( \Gamma(0, r_1, r_2) \), then \( f \) satisfies

\[
\frac{|f|(r_2)}{|f|(r_1)} = \left( \prod_{j=1}^{t} \left( \frac{r_2}{|\alpha_j|} \right)^{m_j} \right) \left( \frac{r_2}{r_1} \right)^q
\]

Corollary II.8.1. Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{H} \) have a set of zeros in \( D \) that consists of a sequence \((\alpha_n)_{n \in \mathbb{N}}\) such that \( \alpha_n \neq 0 \) \( \forall n \in \mathbb{N} \) and where each \( \alpha_n \) is of order \( u_n \). Then \( \|f\|_D = |f(0)| \prod_{n=0}^{\infty} \left( \frac{1}{|\alpha_n|} \right)^{u_n} \).

Corollary II.8.2. Let \( f \in A \) be not quasi-invertible, such that \( f(0) = 1 \) and let \((a_n)_{n \in \mathbb{N}} \) be the sequence of zeros with respective multiplicity \( q_n \). Then the series \( \sum_{n=0}^{\infty} q_n \log \left( \frac{1}{|a_n|} \right) \) converges to \( \log \|f\| \).

By Theorem II.6, II.7, and II.8 we can derive Theorem II.9:

Theorem II.9. A maximal ideal of \( A \) is of infinite codimension if and only if it is coroner.

Proof. Let \( M \) be a maximal ideal of \( A \). By Corollary II.7.2, \( M \) is of the form \( \mathcal{J}(\mathcal{U}) \) with \( \mathcal{U} \) an ultrafilter on \( D \). If \( \mathcal{U} \) is coroner, then \( M \) is not of the form \((x-a)A a \in D\), and by Theorem II.6 it contains no quasi-invertible element and it is of infinite codimension. If \( \mathcal{U} \) is not coroner, either it is a Cauchy filter of limit \( a \in D \) or it is thinner than a circular filter on \( D \) of center \( a \in D \) and diameter \( r \in ]0,1[ \). If \( \mathcal{U} \) is a Cauchy filter, \( M \) is the ideal of functions \( f \) such that \( f(a) = 0 \), then by Theorem II.6., \( M \) is of codimension 1.

Suppose now that If \( \mathcal{U} \) is not a Cauchy filter. Since it is not coroner, it is secant with a disk \( d(a,r) \subset D \) and hence it is thinner than a circular filter \( \mathcal{F} \) of center \( b \in d(a,r) \) and diameter \( s \in ]0,r[ \) so that \( \lim_{\mathcal{F}} |f(x)| = 0 \) \( \forall f \in M \). But since the function \( f \in A \) is an element \( f H(d(a,r)) \), by Theorem 1.9 it cannot satisfy \( \lim_{\mathcal{F}} |f(x)| = 0 \), a contradiction. \( \square \)
3. Araujo semi-norms

Theorem III.1. If $A$ is multbijective, then $\text{Mult}_1(A, \| \cdot \|)$ is dense in $\text{Mult}_m(A, \| \cdot \|)$. 

Proof. By Theorem II.2, each ultrafilter $\mathcal{U}$ on $D$ defines an element $\varphi_{\mathcal{U}}$ of $\text{Mult}(A, \| \cdot \|)$. Conversely, by Corollary II.7.2, every maximal ideal $\mathcal{M}$ is of the form $\mathcal{J}(\mathcal{U})$ with $\mathcal{U}$ an ultrafilter on $D$. Suppose now that $A$ is multbijective and let $\phi \in \text{Mult}_m(A, \| \cdot \|)$. Then $\text{Ker}(\phi)$ is a maximal ideal $\mathcal{M}$ of the form $\mathcal{J}(\mathcal{U})$ with $\mathcal{U}$ an ultrafilter on $D$. Consequently, $\text{Ker}(\varphi_{\mathcal{U}}) = \mathcal{M} = \text{Ker}(\phi)$. But since $A$ is multbijective, then $\phi = \varphi_{\mathcal{U}}$. And then, by Theorem II.1, $\varphi_{\mathcal{U}}$ belongs to the closure of $\text{Mult}_1(A, \| \cdot \|)$ in $\text{Mult}(A, \| \cdot \|)$, which ends the proof. 

On the other hand, consequently to Theorem II.8, we can state Theorem III.: 

Theorem III.2. Let $r, s, R \in [0, +\infty]$ satisfy $0 < r < s < R$ and let $f \in H((0, R))$. Then 

$$\log(|f(s)|) - \log(|f(r)|) \leq \left( \log(|f(R)|) - \log(|f(s)|) \right) \left( \frac{\log(s) - \log(r)}{\log(R) - \log(s)} \right).$$

Proof. Let $q$ be the total number of zeros of $f$ in $d(0, s)$, each counted with its multiplicity. Then by Theorem II.8 we have $\log(|f(s)|) - \log(|f(r)|) \leq q(\log(s) - \log(r))$. On the other hand, $\log(|f(R)|) - \log(|f(s)|) \geq q(\log(R) - \log(s))$. Consequently, 

$$q \leq \frac{\log(|f(R)|) - \log(|f(s)|)}{\log(R) - \log(s)},$$

which ends the proof. 

Theorem III.3. Let $\phi \in \text{Mult}(A, \| \cdot \|)$ and assume that its restriction $\varphi_{\mathcal{F}}$ to $H(D)$ is not $\| \cdot \|$. Then $\phi(f) = \lim_{x \to f} |f(x)| \forall f \in A$. 

Proof. By hypothesis, $\mathcal{F}$ is a circular filter of diameter $l < 1$. Suppose first that $f$ is invertible. Then $|f(x)|$ is a constant $b > 0$. Consequently, $\|f\| = b = \varphi_{\mathcal{F}}(f)$. Suppose $\phi(f) \neq b$. Then $\phi(f) < b$ because $b = \|f\|$. Now consider $h = \frac{1}{f}$. Since $\| \cdot \|_{\mathcal{F}}$ is multiplicative, we see that $\phi(h) > \|h\|$, a contradiction. Consequently, $\phi(f) = \varphi_{\mathcal{F}}(f)$. 

Suppose now $f$ is quasi-invertible. Then $f$ is of the form $Pg$ with $P \in \mathbb{K}[x]$ and $g$ invertible in $A$. Then $\phi(f) = \phi(P)\phi(g) = \varphi_{\mathcal{F}}(P)\varphi_{\mathcal{F}}(g) = \varphi_{\mathcal{F}}(f)$. 

We now suppose that $f$ is not quasi-invertible. By Corollary I.4.1, $f$ has a sequence of zeros $(a_n)_{n \in \mathbb{N}}$ in $D$, each having a multiplicity order $u_n$. By Corollary II.8.1, we have $\lim_{n \to +\infty} |a_n| = 1$, so we can assume $|a_n| \leq |a_{n+1}| \forall n \in \mathbb{N}$. Let $t = \phi(f)$ and $s = \lim_{x \to f} |f(x)|$. We shall show that $t \leq s$.

Suppose first that $\mathcal{F}$ has a disk $d(a, r)$ which contains none of the $a_n$. By Corollary II.8.1 we have $\frac{|f(x)|_{d(a, r)}}{\|f\|} = \prod_{n=1}^{\infty} \left( |a_n - a| \right)^{u_n}$, hence inside the disk $d(a, r)$, $|f(x)|$ is a constant equal to $\|f\| \prod_{n=0}^{\infty} \left( |a_n - a| \right)^{u_n}$ and therefore,
\[ s = \|f\| \prod_{n=1}^{\infty} \left( |a_n - a| \right)^{u_n}. \]  

(1)

For each \( q \in \mathbb{N} \), let \( f_q = \frac{f}{\prod_{n=0}^{q}(x-a_n)^{u_n}} \) and let \( l_q = \sum_{k=0}^{q} u_k \). So, clearly, \( \|f_q\| = \|f\| \quad \forall q \in \mathbb{N} \).

Now, since \( \phi(P) = \phi_{\mathcal{F}}(P) \quad \forall P \in \mathbb{K}[x] \), we have
\[ \phi(\prod_{n=1}^{q}(x-a_n)^{u_n}) = \prod_{n=1}^{q} |a_n - a|^{u_n}, \quad \text{hence} \quad \phi(f_q) = \frac{t}{\prod_{n=1}^{q} |a_n - a|^{u_n}}. \]

But since \( \phi(f_q) \leq \|f_q\| \), that yields
\[ \frac{t}{\prod_{n=1}^{q} |a_n - a|^{u_n}} \leq \|f_q\| = \|f\| \quad \forall q \in \mathbb{N} \]

hence
\[ \frac{t}{\prod_{n=1}^{\infty} |a_n - a|^{u_n}} \leq \|f\| \quad \forall q \in \mathbb{N} \]

Since this is true for every \( q \in \mathbb{N} \), we can derive
\[ t \prod_{n=1}^{\infty} \left( \frac{1}{|a_n - a|^{u_n}} \right) \leq \|f\| \]

hence by (1), \( \frac{t\|f\|}{s} \leq \|f\| \) and therefore \( t \leq s \).

Now consider the case when there exists no disk \( d(a, r) \) belonging to \( \mathcal{F} \), such that none of the \( a_n \) lie in \( d(a, r) \). Since \( \lim_{n \to \infty} |a_n| = 1 \), \( \mathcal{F} \) is a filter admitting a center \( \alpha \). Let \( \rho \) be its diameter: of course \( \rho < 1 \) because \( \phi_{\mathcal{F}} \) is not \( \|\cdot\| \). Consequently, \( d(\alpha, \rho) \) contains finitely many zeros of \( f \alpha_1, \ldots, \alpha_s \) (eventually, if \( \rho = 0 \), then \( d(\alpha, \rho) \) is reduced to the singelton \( \{\alpha\} \).

Suppose first \( \rho = 0 \). Then \( \phi_{\mathcal{F}}(f) = 0 \) and \( \phi(x-\alpha) = 0 \) therefore \( s = t \).

Suppose now \( \rho > 0 \). Suppose \( |a_j - a| \leq \rho \) whenever \( j = 1, \ldots, q \). We can choose \( a \neq a_j \quad \forall j = 1, \ldots, q \). Set \( h = \frac{f}{\prod_{j=1}^{q}(x-a_j)^{u_j}} \). Then \( \phi_{\mathcal{F}}(h) = \frac{s}{\prod_{j=1}^{q} |a_j - \alpha|^{u_j}} \) and \( \phi(h) = \frac{t}{\prod_{j=1}^{q} |a_j - \alpha|^{u_j}}. \) Thus we are led to the same problem with \( h \). Setting
\[ s' = \frac{s}{\prod_{j=1}^{q} |a_j - \alpha|^{u_j}}, \quad t' = \frac{t}{\prod_{j=1}^{q} |a_j - \alpha|^{u_j}}, \]
we have \( t' \leq s' \) hence \( t \leq s \) in all cases and therefore we have proven again that
\[ \phi(h) \leq \phi_{\mathcal{F}}(h) \quad \forall h \in A. \]

(2)

Suppose now that for some \( f \in A \), we have \( \phi(f) < \phi_{\mathcal{F}}(f) \). We can take \( r \in ]r, 1[ \) such that the disk \( d(0, r) \) belongs to \( \mathcal{F} \). Let \( f(x) = \sum_{n=1}^{\infty} b_n x^n \). For every \( q \in \mathbb{N} \), let \( g_q(x) = \sum_{n=1}^{q} b_n x^n \). We notice that when \( q \) is big enough we have
\[ \varphi_f(g_q) = \sup_{n \in \mathbb{N}} |b_n|r^n. \] Set \( w = \sup_{n \in \mathbb{N}} |b_n|r^n. \) Now \( \varphi_f(f - g_q) \leq \sup_{n > q} |b_n|r^n, \) therefore \( \lim_{q \to +\infty} \varphi_f(f - g_q) = 0 \) and hence, by (2), we have
\[
\lim_{q \to +\infty} \phi(f - g_q) = 0. \quad (3)
\]
So, we can take \( q \) such that \( \varphi_f(f - g_q) < \varphi_f(f) \) and hence, by (2), we have \( \phi(f - g_q) < \varphi_f(f). \) But since \( g_q \) is a polynomial, we have \( \phi(g_q) = \varphi_f(g_q) \), hence \( \phi(g_q) > \phi(f). \) Consequently, \( \phi(f - g_q) = \phi(g_q) = w \) when \( q \) is big enough, a contradiction to (3).

By Theorem III.3 we now have the following corollaries:

**Corollary III.3.1.** Let \( F \) be a circular filter on \( D \) of diameter \( r \in ]0, 1[. \) Then \( \varphi_f \) has extension to a norm that belongs to \( \text{Mult}(A, \| \cdot \|). \)

**Proof.** Let \( s \in ]r, 1[ \) and let \( d(a, s) \) be a disk that belongs to \( F \). As an element of \( H(d(a, s)) \), each element \( f \) of \( A \) is such that \( \varphi_f(f) = \lim_{\varphi} |f(x)| \) and that defines a multiplicative norm on \( A \).

**Corollary III.3.2.** Let \( \phi \in \text{Mult}(A, \| \cdot \|) \setminus \text{Mult}_1(A, \| \cdot \|). \) If the restriction of \( \phi \) to \( H(D) \) is of the form \( \varphi_f \) with \( F \) a circular filter on \( D \) of diameter \( r \in ]0, 1[ \), then \( \phi \) is a norm on \( A \).

**Proof.** Indeed, given a disk \( L \) of diameter \( s \in ]r, 1[ \), which belongs to \( F \), \( \varphi_f \) is a norm on \( H(L) \) which contains \( A \).

**Corollary III.3.3.** Let \( \phi \in \text{Mult}(A, \| \cdot \|) \setminus \text{Mult}_1(A, \| \cdot \|). \) If \( \phi \) is not a norm on \( A \), its restriction to \( H(D) \) is \( \| \cdot \| \).

**Proof.** Indeed, if the restriction of \( \phi \) to \( H(D) \) is of the form \( \varphi_f \) with \( F \) a circular filter on \( D \) of diameter \( r \in ]0, 1[ \), then by Corollary III.3.1 \( \phi \) is a norm on \( A \).

The following Theorem III.4 is Theorem 22.33 in [10]:

**Theorem III.4.** Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}(\mathbb{I}K) \) (resp. \( f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}(d(0,r^-)) \)).

All zeros of \( f \) are of order one and the set of zeros of \( f \) is a sequence \( (\alpha_n)_{n \in \mathbb{N}} \) such that \( |\alpha_n| < |\alpha_{n+1}| \) if and only if the sequence \( \frac{a_n}{a_{n+1}} \) is strictly increasing. Moreover, if these properties are satisfied, then the sequence of zeros of \( f \) in \( \mathbb{I}K \) (resp. in \( d(0,r^-) \)) is a sequence \( (\alpha_n)_{n \in \mathbb{N}} \) such that \( \lim_{n \to +\infty} |\alpha_n| = +\infty \) (resp. \( \lim_{n \to +\infty} |\alpha_n| = r \)) and \( |\alpha_n| = \left| \frac{a_n}{a_{n+1}} \right| \).

The following Theorem III.5 is also given in [5] as Theorem 25.5:

**Theorem III.5.** Let \( (a_j)_{j \in \mathbb{N}} \) be a sequence in \( d(0,1^-) \) such that \( 0 < |a_n| \leq |a_{n+1}| \) for every \( n \in \mathbb{N} \) and \( \lim_{n \to +\infty} |a_n| = r \). Let \( (q_j)_{n \in \mathbb{N}} \) be a sequence in \( \mathbb{N}^* \) and let \( B \in [1, +\infty[. \) There exists \( f \in \mathcal{A}(d(0, r^-)) \) satisfying
i) \( f(0) = 1 \)
Corollary III.6.1. Let \( \varepsilon > 0 \) such that \( 0 < |a_n| \leq |a_{n+1}| \) for every \( n \in \mathbb{N} \), \( \lim_{n \to \infty} |a_n| = r \) and let \( (q_n)_{n \in \mathbb{N}} \) be a sequence in \( \mathbb{N}^* \) such that
\[
\prod_{j=0}^{n} \left( \frac{|a_n|}{r} \right)^{q_j} > 0.
\]

Let \( B \in ]1, +\infty[\). There exists \( f \in A \) satisfying

i) \( f(0) = 1 \)

ii) \( \|f\| \leq B \prod_{j=0}^{\infty} \left( \frac{r}{|a_n|} \right)^{q_j} \) whenever \( n \in \mathbb{N} \)

iii) for each \( n \in \mathbb{N} \), \( a_n \) is a zero of \( f \) of order \( z_n \geq q_n \).

Notation. Let \( (a_n)_{n \in \mathbb{N}} \) a sequence in \( D \) such that \( |a_n| \leq |a_{n+1}| \) and \( \lim_{n \to +\infty} |a_n| = 1 \) and let \( (q_n)_{n \in \mathbb{N}} \) be a sequence of integers \( (q_n \geq 0) \). The family \( (a_n, q_n)_{n \in \mathbb{N}} \) is called a divisor of \( D \). The definition applies to a divisor where all \( q_n \) are null but finitely many.

The family of divisors of \( D \) is provided with a natural order: given two divisors \( T = (a_n, q_n)_{n \in \mathbb{N}} \) and \( E = (a_n, s_n)_{n \in \mathbb{N}} \), we say that \( T \leq E \) if \( q_n \leq s_n \) \( \forall n \in \mathbb{N} \).

Let \( f \in \mathcal{H} \) and let \( (a_n, q_n)_{n \in \mathbb{N}} \) be the set of zeros of \( f \), each zero \( a_n \) being of order \( q_n \). We will denote by \( \mathcal{T}(f) \) this family \( (a_n, q_n)_{n \in \mathbb{N}} \) and the expression \( \mathcal{T}(f) \) is then called divisor of \( f \).

An interesting question was whether certain elements of \( \text{Mult}(A, \| . \|. \) \) may have a kernel that is neither null nor a maximal ideal. The question was solved by Jesus Araujo thanks to this nice example [1].

In the proof of Theorem III.7, we will need the following Theorem III.6 that comes from Theorems 28.14 and 29.6 in [10].

Theorem III.6. Let \( E = (a_n, q_n)_{n \in \mathbb{N}} \) be a divisor on \( D \) with \( a_n \neq 0 \) \( \forall n \in \mathbb{N} \) and let \( \varepsilon > 0 \). There exists \( f \in \mathcal{H} \) such that \( \mathcal{T}(f) \geq E, f(0) = 1 \) and \( |f(r)| \leq |E|(r)(1 + \varepsilon) \) \( \forall r \in [0, 1[\). Moreover, if \( \mathbb{K} \) is spherically complete, then there exists \( f \in B \) such that \( \mathcal{T}(f) = E \).

Corollary III.6.1. Let \( \mathbb{K} \) be spherically complete. Let \( (a_j)_{j \in \mathbb{N}} \) be a coroner sequence such that \( \prod_{n=0}^{\infty} |a_n| > 0 \). There exists \( f \in \mathcal{H} \) admitting each \( a_n \) as a zero of order 1 and having no other zeros.

Corollary III.6.2. Let \( f, g \in A \) be such that \( \mathcal{T}(g) \leq \mathcal{T}(f) \). There exists \( h \in A \) such that \( f = gh \).

Theorem III.7. (J. Araujo) Let \( h(x) = \sum_{n=0}^{\infty} a_n x^n \) and suppose that the sequence \( \left( \frac{|a_n|}{|a_{n+1}|} \right)_{n \in \mathbb{N}} \) is strictly increasing, of limit 1. Then \( h \) belongs to \( A \). Moreover,
putting \( r_n = \frac{|a_n|}{|a_{n+1}|}, n \in \mathbb{N} \), \( h \) admits a unique zero on each circle \( C(0, r_n) \) and has no other zero in \( D \).

Let \( \mathcal{N} \) be an ultrafilter on \( \mathbb{N} \) and for every \( f \in A \), let \( \phi(f, n) = \|f\|_{d(a_n, r)} \). Let \( \varphi_r(f) = \lim_{\mathcal{N}} \phi(f, n) \).

Then \( \varphi_r \) belongs to \( \text{Mult}(A, \| . \|) \) and \( \text{Ker}(\varphi_r) \) is neither null nor a maximal ideal of \( A \). Moreover, \( \text{Ker}(\varphi_r) \) does not depend on \( r \in ]0, 1[ \).

However each so defined semi-norm \( \varphi_r \) belongs to the closure of \( \text{Mult}_1(A, \| . \|) \) in \( \text{Mult}(A, \| . \|) \).

Proof. \( h \) belongs to \( A \) because the sequence \( (a_n) \) is bounded. Next, \( h \) has a unique zero \( \alpha_n \) in each circle \( C(0, r_n) \) and no other zero in \( D \) by Theorem III.4.

Let \( \mathcal{M} \) be the ideal of the \( f \in A \) such that \( \lim_{\mathcal{N}} |f(\alpha_n)| = 0 \). Of course \( h \) belongs to \( \mathcal{M} \) and \( \text{Ker}(\varphi_r) \) is strictly included in \( \mathcal{M} \). Indeed, since \( h \) admits a unique zero in the disk \( d(\alpha_n, r) \), it satisfies \( \|h\|_{d(\alpha_n, r)} = |h|(r_n) \frac{r_n}{r} \) and therefore \( \lim_{\mathcal{N}} \|h\|_{d(\alpha_n, r)} = \frac{1}{r} \), which proves that \( h \) does not belong to \( \text{Ker}(\varphi_r) \).

On the other hand, we will prove that \( \text{Ker}(\varphi_r) \) is not null. Let \( (q_n)_{n \in \mathbb{N}^*} \) be a sequence of positive integers satisfying \( q_n \leq q_{n+1} \forall n \in \mathbb{N}^* \), \( \lim_{n \to +\infty} q_n = +\infty \) and such that the series \( \sum_{n=1}^{+\infty} q_n \log\left(\frac{1}{r_n}\right) \) converges: we can easily find the sequence \( (q_n) \) since \( \lim_{n \to +\infty} r_n = 1 \). Now, consider the divisor \( (\alpha_n, q_n)_{n \in \mathbb{N}} \) of \( D \). By Theorem III.6 there exists \( g \in A \) admitting each \( \alpha_n \) as a zero of order \( t_n \geq q_n \) and such that \( |g|(r_n) \leq |T|(r_n) + 1 \forall n \in \mathbb{N}^* \). Consequently, \( g \) is bounded in \( D \) and hence belongs to \( A \). Next, for every \( n \in \mathbb{N}^* \), by Corollary II.8.1 we have

\[
\|g\|_{d(\alpha_n, r)} \leq |g|(r_n) \left( \frac{r_n}{r} \right)^{t_n} \leq \|g\| \left( \frac{r}{r_n} \right)^{q_n}.
\]

Since the sequence \( (q_n)_{n \in \mathbb{N}^*} \) tends to \( +\infty \) and the sequence \( (r_n) \) is increasing, we have \( \lim_{n \to +\infty} \|g\|_{d(\alpha_n, r)} = 0 \), which proves that \( g \) belongs to \( \text{Ker}(\varphi_r) \).

Let \( f \in \text{Ker}(\varphi_r) \) and let \( s \in ]0, 1[ \). If \( s \leq r \), it is obvious that \( f \) belongs to \( \text{Ker}(\varphi_s) \).

Now suppose \( s > r \). Consider an element \( L \) of \( \mathcal{N} \) such that \( \inf_{n \in L} \|f\|_{d(\alpha_n, s)} = 0 \).

We will prove that \( \inf_{n \in L} \|f\|_{d(\alpha_n, s)} = 0 \).

For each \( n \in \mathbb{N}^* \), by Theorem III.2, we have

\[
\log\left(\|f\|_{d(\alpha_n, s)}\right) - \log\left(\|f\|_{d(\alpha_n, r)}\right) \leq \left( \log\left(\|f\|_{d(\alpha_n, r_n)}\right) - \log\left(\|f\|_{d(\alpha_n, s)}\right) \right) \frac{\log(s) - \log(r)}{\log(r_n) - \log(s)}.
\]

Suppose that the sequence \( \|f\|_{d(\alpha_n, s)} \) does not tend to 0.
There exists a sequence \((u_n)_{m \in \mathbb{N}}\) of \(\mathbb{N}^*\) such that \(\|f\|_{d(u_m, s)} > b, \forall m \in \mathbb{N}\) with \(b > 0\). But then, we get to a contradiction with (1).

Consequently, \(\inf_{n \in \mathbb{N}} \|f\|_{d(u_n, s)} = 0\) and therefore \(f\) belongs to \(\text{Ker}(\varphi_s)\), which proves that \(\text{Ker}(\varphi_s) = \text{Ker}(\varphi_r)\).

Consider now a neighborhood \(\mathcal{V}(\varphi_r, f_1, \ldots, f_q, \varepsilon)\) of \(\varphi_r\), where \(f_1, \ldots, f_q \in A\) and \(\varepsilon > 0\), with respect to the topology of pointwise convergence, i.e.
\[
\mathcal{V}(\varphi_r, f_1, \ldots, f_q, \varepsilon) = \{ \varphi \in \text{Mult}(A, \| \cdot \|) \mid \| \varphi(f_j) - \varphi(f_j) \|_\infty \leq \varepsilon, \quad j = 1, \ldots, q, q \in \mathbb{N}^* \}.
\]
By definition of that topology, there exists a subset \(G\) of \(\mathbb{N}\) such that \(|\varphi_r(f_j) - \| f_j \|_{d(a_n, r)}|_\infty \leq \varepsilon \forall n \in G, \forall j = 1, \ldots, q\). But now, in each disk \(d(a_n, r)\), we can take a class \(d(b_n, r^-)\) where none of the \(f_j\) admits a zero, and hence we have \(|f_j(b_n)| = \| f_j \|_{d(a_n, r)} \forall j = 1, \ldots, q\) and hence \(|\varphi(b_n(f_j) - \varphi_r(f_j)|_\infty \leq \varepsilon \forall j = 1, \ldots, q\). Consequently, \(\varphi(b_n)\) belongs to \(\mathcal{V}(\varphi_r, f_1, \ldots, f_q, \varepsilon)\), which proves that \(\varphi_r\) lies in the closure of \(\text{Mult}_1(A, \| \cdot \|)\) and that finishes the proof of Theorem III.7. \(\square\)

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REFERENCES


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GENERALIZED HEAT EQUATION UNDER CONFORM DERIVATIVE

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ABSTRACT. In the present work, we establish the existence and uniqueness result of the linear heat equation with Conform derivative in Colombeau generalized algebra. We using for the first time the notion of a generalized conformable semigroup and the purpose of introducing Conform derivative is regularizing it in Colombeau.

1. INTRODUCTION

In this paper the heat equation is considered with the first order time derivative changed to a conform derivative. for each type of data we always ask what is the optimal corresponding nonlocal model to be applied. Moreover, many authors studied Conform operators with local, nonlocal, singular and non-singular kernels [2]. The Riemann and Caputo fractional calculus may not provide us the required kernel in order to extract important information from these types of systems. At this stage, we ask the following question. Can we generalize the standard fractional Riemann-Liouville integrals in a way such that we obtain unification to Riemann-Liouville, Hadamard and other fractional derivatives. The importance of this procedure is to decide which differentiation operator should be used as a starting point for the iteration procedure. For the standard fractional calculus, we iterate the usual integral of a function and using the Cauchy formula we obtain the integral of higher integer orders and then replace this integer by any real number. In [1] the author presented and developed the definitions of Conform derivative and set the basic concepts in this new simple interesting fractional calculus. The fractional versions of chain rule, exponential functions, Gronwall’s inequality, integration by parts, Taylor power series expansions, Laplace transforms and linear differential systems are proposed and discussed. The present paper concerns the study of existence and uniqueness to equation with Conform differentiation in extended Colombeau algebra. We consider Conform differentiation for indicating to existence and uniqueness heat equation in extended Colombeau algebra. The

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reason for introducing conform derivatives into algebra of generalized functions was the possibility of solving nonlinear problems with singularities and derivatives of arbitrary real order in it. We use specific space of Colombeau algebra type in order to give a sense of our problem. Colombeau algebras is a differential algebra, commutative, associative in which we can inject $D'$ the set of distributions so that the product of smooth functions and the usual derivative of distributions are respected [5], [6], [7]. From the previous ideas we will discuss the existence and uniqueness of such equation in a specific spaces coincide with the usual spaces when $\alpha \to 1$.

The paper is organized as follows: After this introduction, we present some concepts concerning the Colombeau’s algebra. In section 3 we give the definitions and we prove some properties concerning the Conform derivative. The embedding of this derivation in Colombeau algebra takes place in section 4. In the last section we discuss the existence and uniqueness of our problem.

2. PRELIMINAIRES

We shall fix the notation and introduce a number of known as well as new classes of generalized functions here. For more details, see [7].

Let $\Omega$ be an open subset of $\mathbb{R}^n$. The basic objects of the theory as we use it are families $(u_\varepsilon)_{\varepsilon \in (0,1]}$ of smooth functions $u_\varepsilon \in C^\infty(\Omega)$ for $0 < \varepsilon \leq 1$. We single out the following subalgebra.

Moderate families, denoted by $E_M(\Omega)$, are defined by the property :

\[
\forall K \subset \Omega, \forall \alpha \in \mathbb{N}^n_0, \exists p \geq 0 : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O_{\varepsilon \to 0}(\varepsilon^{-p})
\]

Null families, denoted by $E_M(\Omega)$, are defined by the property :

\[
\forall K \subset \Omega, \forall \alpha \in \mathbb{N}^n_0, \forall q \geq 0 : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O_{\varepsilon \to 0}(\varepsilon^q)
\]

Thus moderate families satisfy a locally uniform polynomial estimate as $\varepsilon \to 0$, together with all derivatives, while null functions vanish faster than any power of $\varepsilon$ in the same situation. The null families form a differential ideal in the collection of moderate families.

The Colombeau algebra is the factor algebra

\[
\mathcal{G}(\Omega) = E_M(\Omega)/\mathcal{N}(\Omega)
\]

The algebra $\mathcal{G}(\Omega)$ just defined coincides with the special Colombeau algebra in [5], where the notation $\mathcal{G}^s(\Omega)$ has been employed. It was called the simplified Colombeau algebra in [5].

The Colombeau algebra on a closed half space $\mathbb{R}^n \times [0, 1)$ is defined in a similar way. The restriction of an element $u \in \mathcal{G}(\mathbb{R} \times [0, 1))$ to the line $\{t = 0\}$ is defined on representatives by

\[
u/\{t = 0\} = \text{Class of } (u_\varepsilon(\cdot, 0))_{\varepsilon \in (0,1]}
\]
Similarly, restrictions of the elements of $G(\Omega)$ to open subsets of $\Omega$ are defined on representatives. One can see that $\Omega \longrightarrow G(\Omega)$ is a sheaf of differential algebras on $\mathbb{R}^n$. The space of compactly supported distributions is imbedded in $G(\Omega)$ by convolution:

$$i : \left\{ \begin{array}{l}
\mathcal{E}'(\Omega) \longrightarrow G(\Omega) \\
o \rightarrow i(\omega) = \text{class of } (\omega \ast (\phi_\varepsilon)/\Omega)_{\varepsilon \in (0,1]}
\end{array} \right.$$ 

where

$$\phi_\varepsilon(x) = \varepsilon^{-n} \phi\left(\frac{x}{\varepsilon}\right)$$

is obtained by scaling a fixed test function $\mathcal{S}(\mathbb{R}^n)$ of integral one with all moments vanishing. By the sheaf property, this can be extended in a unique way to an imbedding of the space of distributions $\mathcal{D}(\Omega)$.

One of the main features of the Colombeau construction is the fact that this imbedding renders $C^\infty(\Omega)$ a faithful subalgebra. In fact, given $f \in C^\infty(\Omega)$, one can define a corresponding element of $G(\Omega)$ by the constant imbedding $\sigma(f) = \text{class of } \{(\varepsilon, x) \longrightarrow f(x)\}$. Then the important equality $i(f) = \sigma(f)$ holds in $G(\Omega)$.

If $u \in G(\Omega)$ and $f$ is a smooth function which is of at most polynomial growth at infinity, together with all its derivatives, the superposition $f(u)$ is a well-defined element of $G(\Omega)$.

We need a couple of further notions from the theory of Colombeau generalized functions. An element $u$ of $G(\Omega)$ is called of local $L^p$-type ($1 \leq p \leq \infty$), if it has a representative with the property $\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^p(K)}$ exists for every $K \subset \Omega$.

Regularity theory is based on the subalgebra $G^\infty(\Omega)$ of regular generalized functions in $G(\Omega)$. It is defined by those elements which have a representative satisfying

$$\forall K \subset \Omega, \forall \alpha \in \mathbb{N}_0^n, \exists \rho \geq 0 : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O_{\varepsilon \to 0}(\varepsilon^{-\rho})$$

Observe the change of quantifiers with respect to formula 2; locally, all derivatives of a regular generalized function have the same order of growth in $\varepsilon > 0$. One has that (see [6]).

$$G^\infty(\Omega) \cap \mathcal{D}'(\Omega) = C^\infty(\Omega)$$

For the purpose of describing the regularity of Colombeau generalized functions, $G^\infty(\Omega)$ plays the same role as $C^\infty(\Omega)$ does in the setting of distributions.

A net $(r_\varepsilon)_{\varepsilon \in (0,1]}$ of complex numbers is called a slow scale net if

$$|r_\varepsilon| = O_{\varepsilon \to 0}(\varepsilon^{-\rho})$$

for every $\rho \geq 0$. We refer to [6] for a detailed discussion of slow scale nets. Finally, an element $u \in G(\Omega)$ is called of total slow scale type, if for some representative, $\|\partial^\alpha u_\varepsilon\|_{L^\infty(K)}$ forms a slow scale net for every $K \subset \Omega$ and $\alpha \in \mathbb{N}_0^n$.

We end this section by recalling the association relation on the Colombeau algebra $G(\Omega)$. It identifies elements of $u \in G(\Omega)$ if they coincide in the weak limit. That
is, $u, v \in G(\Omega)$ are called associated,

$$u \approx v, \text{ if } \lim_{\varepsilon \to 0} \int (u_\varepsilon(x) - v_\varepsilon(x)) \psi(x) dx = 0$$

3. Conformable derivative

In this section we will give some definition and properties concerning the new derivative important in the following.

**Definition 3.1.** [11] Let $\alpha \in (n, n+1]$ and $f : [0, \infty) \to \mathbb{R}$ be $n$-differentiable at $t > 0$, then the conformable derivative of $f$ of order $\alpha$ is defined by

$$f^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{f^{(n)}(t + \varepsilon t^{n+1-\alpha}) - f^{(n)}(t)}{\varepsilon}$$

$$f^{(\alpha)}(0) = \lim_{t \to 0} f^{(\alpha)}(t)$$

**Remark 3.1.** [11] As consequence of the previous definition, one can easily show that

$$f^{(\alpha)}(t) = t^{n+1-\alpha} f^{(n+1)}(t)$$

where $\alpha \in (n, n+1]$, and $f$ is $(n+1)$-differentiable at $t > 0$.

In [2] we find the following proposition.

**Proposition 3.1.** [2] Let $f, g$ two function $\alpha$-derivatives and $a, b \in \mathbb{R}$. We have the following properties.

1. $(a f + b g)^{(\alpha)} = a f^{(\alpha)} + b g^{(\alpha)}$,
2. $(f g)^{(\alpha)} = f^{(\alpha)} g + f g^{(\alpha)}$,
3. $(t^p)^{(\alpha)} = p t^{p-\alpha}$,
4. $(\frac{f}{g})^{(\alpha)} = \frac{f^{(\alpha)} g - f g^{(\alpha)}}{g^2}$,
5. If $c \in \mathbb{R}$, $c^{(\alpha)} = 0$.

**Proposition 3.2.** If $x$ is a continuous map, then $t \to x^{(\alpha)}(t)$ is a continuous map.

**Proof.** Since $x$ is a continuous map $t \in \mathbb{R}_+^* \to x(t + \varepsilon t^{1-\alpha})$ is a continuous, thus $\forall \beta > 0, \exists \alpha > 0,$

$$\frac{| x(t + \varepsilon t^{1-\alpha}) - x(t_0 + \varepsilon t_0^{1-\alpha}) |}{\varepsilon} \leq \beta$$

whenever $|t - t_0| \leq \alpha$, by passing to limite $\varepsilon \to 0$ we get $|x^{(\alpha)}(t) - x^{(\alpha)}(t_0)| \leq \beta$ as desired.

**Proposition 3.3.** Let $f : X \to X$, be a Lipschitz map. i.e.

$$|f(x) - f(y)| \leq k|x - y|, \forall x, y \in X \text{ and } k \in ]0, 1[.$$ The problem of Cauchy

$$\begin{cases} x^{(\alpha)}(t) = f(x(t)), & t > 0 \\ x(0) = x_0 \end{cases}$$

has a unique solution.
Proof. By 5.3 $x$ is continuous, the sequence $x_{n+1} = f(x_n)$ is a Cauchy’s sequence, since $\mathbb{R}$ is a complete space then $x_n$ converge to the unique solution of (3.3) $\square$

**Definition 3.2.** [11] Let $\alpha \in (n, n+1]$. We define the $\alpha$-integral by:

$$(I^\alpha f)(t) = \int_0^t s^{\alpha-n} f(s) ds.$$  

According to the previous writing we set

$$dx_\alpha = \frac{dx}{x^{\alpha-1}}$$

**Theorem 3.1.** [11] We have the following inequality

$$(I^\alpha f)^{(\alpha)}(t) = f(t).$$

for $t \geq 0$

**Example 3.1.** It is easy to prove that:

$$I^\alpha(\sin(t)) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+\alpha}}{(2n+\alpha)(2n+1)!}$$

where $\alpha \in (1, 2)$

**Definition 3.3.** [9] Let $\alpha \in (0, 1)$. For a Banach space $X$. A family $\{T(t)\}_{t \geq 0} \subset L(X, X)$ is called a conform $\alpha$-semigroup if:

1. $T(0) = I$
2. $T\left(\left(s + \frac{t}{\alpha}\right)^{\frac{1}{\alpha}}\right) = T(\frac{s}{\alpha}) T\left(\frac{t}{\alpha}\right)$, for all $s, t \in [0, \infty)$

**Example 3.2.** Let $A$ be a bounded linear operator on $X$. Define $T(t) = e^{2\sqrt{t}A}$. Then $\{T(t)\}_{t \geq 0}$ is a $\frac{1}{2}$-semigroup. Indeed:

1. $T(0) = e^{0A} = I$
2. $\forall s, t \in [0, \infty)$, $T\left(\left(s + t\right)^{2}\right) = e^{2(t+s)A} = e^{2tA} e^{2sA} = T(s) T(t)$

**Definition 3.4.** [9] An $\alpha$-semigroup $T(t)$ is called a $C_0$-semigroup if, for each fixed $x \in X$, $T(t)x \rightarrow x$ as $t \rightarrow 0^+$

The conformable $\alpha$-derivative of $T(t)$ at $t = 0$ is called the $\alpha$-infinitesimal generator of the conform $\alpha$-semigroup $T(t)$, with domain equals

$$\{x \in X, \lim_{t \rightarrow 0} T(t)x \text{ exist }\}$$

In the sequel $\alpha \in (0, 1)$.

4. **Imbedding of the conformable differentiation into extended Colombeau algebra of generalized functions**

Let $u_\varepsilon(x)$ represents a Colombeau generalized function $u \in G^\varepsilon(\mathbb{R})$: The conformable derivative for $0 < \alpha < 1$ is defined by:

$$D^\alpha u_\varepsilon(x) = x^{1-\alpha} u_\varepsilon'(x)$$
we have
\[
| D^\alpha u_\varepsilon(x) | = | x^{1-\alpha} u_\varepsilon'(x) | \\
| D^\alpha u_\varepsilon(x) | \leq | x^{1-\alpha} | | u_\varepsilon'(x) |
\]
so,
\[
\sup_{x \in K} | D^\alpha u_\varepsilon(x) | \leq a^{1-\alpha} \sup_{x \in K} | u_\varepsilon'(x) |
\]
we use the regularization for \( 1 < \alpha < 1 \)
\[
\tilde{D}^\alpha u_\varepsilon(x) = \int_\mathbb{R} D^\alpha u_\varepsilon(s) \phi_\varepsilon(x-s) ds
\]
The convolution form is given by:
\[
\tilde{D}^\alpha u_\varepsilon(x) = D^\alpha u_\varepsilon * \phi_\varepsilon(x)
\]
we indicate that \( | \tilde{D}^\alpha u_\varepsilon(x) - D^\alpha u_\varepsilon(x) | \approx 0 \)
\[
| \tilde{D}^\alpha u_\varepsilon(x) - D^\alpha u_\varepsilon(x) | = | D^\alpha u_\varepsilon * \phi_\varepsilon(x) - D^\alpha u_\varepsilon(x) | \\
| \tilde{D}^\alpha u_\varepsilon(x) - D^\alpha u_\varepsilon(x) | = | D^\alpha u_\varepsilon * \phi_\varepsilon(x) - D^\alpha u_\varepsilon * \delta(x) | \\
| \tilde{D}^\alpha u_\varepsilon(x) - D^\alpha u_\varepsilon(x) | = | D^\alpha u_\varepsilon * (\phi_\varepsilon(x) - \delta(x)) | \\
\]
\[
\tilde{D}^\alpha u_\varepsilon(x) - D^\alpha u_\varepsilon(x) = \int_\mathbb{R} | D^\alpha u_\varepsilon(x-s) | | \phi_\varepsilon(s) - \delta(s) | ds \rightarrow 0
\]
as \( \varepsilon \rightarrow 0 \). Since \( \lim | \phi_\varepsilon(s) - \delta(s) | \), then
\[
\tilde{D}^\alpha u_\varepsilon(x) \approx D^\alpha u_\varepsilon(x)
\]
Using the fact that \( \phi_\varepsilon(x) \) has the compact support on \( K_0 \), so by Holder inequalities, have the following calculations:
we have
\[
\tilde{D}^\alpha u_\varepsilon(x) = \int_\mathbb{R} D^\alpha u_\varepsilon(x-s) \phi_\varepsilon(s) ds
\]
\[
| \tilde{D}^\alpha u_\varepsilon(x) | = | \int_\mathbb{R} D^\alpha u_\varepsilon(x-s) \phi_\varepsilon(s) ds | = | \int_{K_0} D^\alpha u_\varepsilon(x-s) \phi_\varepsilon(s) ds |
\]
\[
| \tilde{D}^\alpha u_\varepsilon(x) | = \int_{K_0} | D^\alpha u_\varepsilon(x-s) | | \phi_\varepsilon(s) | ds
\]
\[
\sup_{x \in K} | \tilde{D}^\alpha u_\varepsilon(x) | = \sup_{x \in K} \int_{K_0} | D^\alpha u_\varepsilon(x-s) | | \phi_\varepsilon(s) | ds
\]
so,
\[
\sup_{x \in K} | \tilde{D}^\alpha u_\varepsilon(x) | \leq \sup_{x \in K_1} | D^\alpha u_\varepsilon(x) | \int_{K_0} | \phi_\varepsilon(s) | ds
\]
\[
\sup_{x \in K} | \tilde{D}^\alpha u_\varepsilon(x) | \leq C_1 \varepsilon^p
\]
and
\[
\frac{d}{dx}(\tilde{D}^\alpha u_\varepsilon(x)) = \frac{d}{dx}(D^\alpha u_\varepsilon) * \phi_\varepsilon(x) = D^\alpha u_\varepsilon * \frac{d}{dx}(\phi_\varepsilon(x))
\]
then,
\[
\sup_{x \in K} \left| \frac{d}{dx}(\tilde{D}^\alpha u_\varepsilon(x)) \right| \leq \sup_{x \in K_1} \left| D^\alpha u_\varepsilon(x) \right| \int_{K_0} \left| \frac{d}{ds}(\phi_\varepsilon(s)) \right| ds \leq C_2 \varepsilon^p
\]
In order to prove moderateness for higher derivatives a similar calculation is applied.
\[
\sup_{x \in K} \left| \partial^n \tilde{D}^\alpha u_\varepsilon(x) \right| \leq C_3 \varepsilon^p
\]

5. Generalized Conformable Semigroup

We define
\[
\mathcal{E}_M(\mathbb{R}) := \left\{ (x_\varepsilon)_{\varepsilon} \in (\mathbb{R})^{(0,1)} / \exists m \in \mathbb{N}, |x_\varepsilon| = O_{\varepsilon \to 0}(\varepsilon^{-m}) \right\}
\]
and
\[
\mathcal{N}(\mathbb{R}) := \left\{ (x_\varepsilon)_{\varepsilon} \in (\mathbb{R})^{(0,1)} / \forall m \in \mathbb{N}, |x_\varepsilon| = O_{\varepsilon \to 0}(\varepsilon^m) \right\}
\]
It is easy to prove that

**Proposition 5.1.** The space \( \mathcal{E}_M(\mathbb{R}) \) is an algebra and \( \mathcal{N}(\mathbb{R}) \) ideal in \( \mathcal{E}_M(\mathbb{R}) \)

**Definition 5.1.** We define the Colombeau algebra type by:
\[
\mathcal{R} = \mathcal{E}_M(\mathbb{R}) / \mathcal{N}(\mathbb{R})
\]

**5.1. Locally convex and complete spaces**

**Definition 5.2.** Let \( X \) be a vector space with a seminorms family \( (p_i)_{i \in I} \). If \( \tau_i \) is the topology defined by the only semi-norm \( p_i \). If \( \tau \) is the super bound of topology \( \tau_i \). The space provided with this topology \( \tau \) is called a locally convex space

A basis of 0-neighbourhood is the set of all "balls" of the seminorms \( (p_i)_{i \in I} \)
\[
B(i,r) = \left\{ x \in X / \ p_i(x) < r \right\}, \ \forall i \in I \text{ and } r > 0.
\]
Then, \( (x_n)_{n \in \mathbb{N}} \) is a Cauchy sequence iff
\[
(\forall \varepsilon > 0) \ (\forall i \in I) \ (\exists n_0 \in \mathbb{N}) \ (\forall n, p \in \mathbb{N} \text{ if } n \geq n_0 \Rightarrow p_i(x_{n+p} - x_n) < \varepsilon)
\]

**Definition 5.3.** We said that \( \mathcal{D} \) is dense in locally convex space \( X \) iff
\[
(\forall x \in X) \ (\exists y \in \mathcal{D}) \ (\forall \varepsilon > 0) \ (\forall i \in I) \text{ we have } p_i(x - y) < \varepsilon
\]
and \( X \) is sequentially complete if any Cauchy sequence converges to an element \( e \) in \( X \).
5.2. Generalized semigroup

Definition 5.4. Let \( X \) be a locally convex space with a seminorm family \((p_i)_{i \in I}\). We define

\[
E_M(X) := \left\{ (x_\varepsilon)_{\varepsilon} \in (X)^{(0,1)} / \exists m \in \mathbb{N}, \forall i \in I, p_i(x_\varepsilon) = O_{\varepsilon \to 0}(\varepsilon^{-m}) \right\}
\]

and

\[
\mathcal{N}(X) := \left\{ (x_\varepsilon)_{\varepsilon} \in (X)^{(0,1)} / \forall m \in \mathbb{N}, \forall i \in I, p_i(x_\varepsilon) = O_{\varepsilon \to 0}(\varepsilon^m) \right\}
\]

We define the Colombeau algebra type by:

\[
\tilde{X} = E_M(X) / N_s(X)
\]

First, we are looking if it is possible to define a map \( A : \tilde{X} \to \tilde{X} \) by means of a given family \((A_\varepsilon)_{\varepsilon \in (0,1)}\) of maps \( A_\varepsilon : X \to X \) where \( A_\varepsilon \) is a linear and continuous operator. The general requirement is given in the following

Lemma 5.1. Let \((A_\varepsilon)_{\varepsilon \in (0,1)}\) be a given family of maps \( A_\varepsilon : X \to X \). For each \((x_\varepsilon)_{\varepsilon} \in E_M(X)\) and \((y_\varepsilon)_{\varepsilon} \in \mathcal{N}(X)\), suppose that

(1) \[
\left[ A_\varepsilon x_\varepsilon \right]_{\varepsilon} \in E_M(X)
\]

(2) \[
\left( A_\varepsilon(x_\varepsilon + y_\varepsilon) \right)_{\varepsilon} - \left( A_\varepsilon x_\varepsilon \right)_{\varepsilon} \in \mathcal{N}(X).
\]

Then

\[
A : \tilde{X} \to \tilde{X}
\]

is well defined.

Proof. From the first property we see that the class \( \left[ (A_\varepsilon x_\varepsilon)_{\varepsilon} \right] \in \tilde{X} \)

Let \( x_\varepsilon + x_\varepsilon \) be another representative of \( x = [x_\varepsilon] \)

From the second property we have

\[
\left( A_\varepsilon(x_\varepsilon + y_\varepsilon) \right)_{\varepsilon} - \left( A_\varepsilon x_\varepsilon \right)_{\varepsilon} \in \mathcal{N}(X)
\]

and

\[
\left[ (A_\varepsilon(x_\varepsilon + y_\varepsilon))_{\varepsilon} \right] = \left[ (A_\varepsilon x_\varepsilon))_{\varepsilon} \right] \text{ in } \tilde{X}
\]

Then \( A \) is well defined.

Definition 5.5. Let \( S E_M(\mathbb{R}_+: L_c(X)) \) is the space of nets \( (S_\varepsilon)_{\varepsilon} \) of strongly continuous mappings \( S_\varepsilon : \mathbb{R}_+ \to L_c(X) \), \( \varepsilon \in (0,1) \) with the property that for every \( T > 0 \) there exists \( a \in \mathbb{R} \) such that

\[
\sup_{t \in [0,T]} \| S_\varepsilon(t^{1/2}) \| = O_{\varepsilon \to 0}(\varepsilon^a),
\]

and \( S N(\mathbb{R}_+: L_c(X)) \) is the space of nets \( (N_\varepsilon)_{\varepsilon} \) of strongly continuous mappings \( N_\varepsilon : \mathbb{R}_+ \to L_c(X) \), \( \varepsilon \in (0,1) \) with the properties:
For every $b \in \mathbb{R}$ and $T > 0$
\[
\sup_{t \in [0,T]} \| N_\varepsilon(t^{\frac{1}{\alpha}}) \| = O_{\varepsilon \to 0}(\varepsilon^b),
\]
(5.2)

There exist $t_0 > 0$ and $a \in \mathbb{R}$ such that
\[
\sup_{t < t_0} \| \frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t} \| = O_{\varepsilon \to 0}(\varepsilon^a),
\]
(5.3)

There exists a net $(H_\varepsilon)_\varepsilon$ in $\mathcal{L}_c(X)$ and $\varepsilon_0 \in (0,1)$ such that
\[
\lim_{t \to 0} \frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t} H_\varepsilon x, \ x \in X,
\]
(5.4)

For every $b > 0$,
\[
\| H_\varepsilon \| = O_{\varepsilon \to 0}(\varepsilon^b),
\]
(5.5)

**Proposition 5.2.** $\mathcal{SE}_M(\mathbb{R}_+: \mathcal{L}_c(X))$ is algebra with respect to composition and $\mathcal{SN}(\mathbb{R}_+: \mathcal{L}_c(X))$ is an ideal of $\mathcal{SE}_M(\mathbb{R}_+: \mathcal{L}_c(X))$

**Proof.** Let $(S_\varepsilon(t^{\frac{1}{\alpha}}))_\varepsilon \in \mathcal{SE}_M(\mathbb{R}_+: \mathcal{L}_c(X))$ and $(N_\varepsilon(t^{\frac{1}{\alpha}}))_\varepsilon \in \mathcal{SN}_M(\mathbb{R}_+: \mathcal{L}_c(X))$

We will prove only the second assertion, i.e., That
\[
(S_\varepsilon(t^{\frac{1}{\alpha}})N_\varepsilon(t^{\frac{1}{\alpha}}))_\varepsilon, (N_\varepsilon(t^{\frac{1}{\alpha}})S_\varepsilon(t^{\frac{1}{\alpha}}))_\varepsilon \in \mathcal{SN}_M(\mathbb{R}_+: \mathcal{L}_c(X))
\]
where $S_\varepsilon(t^{\frac{1}{\alpha}})N_\varepsilon(t^{\frac{1}{\alpha}})$ denotes the composition.

Let $\varepsilon \in (0,1).$ By (2) and (3), for some $a \in \mathbb{R}$ and every $b \in \mathbb{R},$
\[
\| S_\varepsilon(t^{\frac{1}{\alpha}})N_\varepsilon(t^{\frac{1}{\alpha}}) \| \leq \| S_\varepsilon(t^{\frac{1}{\alpha}}) \| \| N_\varepsilon(t^{\frac{1}{\alpha}}) \| = O_{\varepsilon \to 0}(\varepsilon^{a+b}),
\]

The same holds for $\| N_\varepsilon(t^{\frac{1}{\alpha}})S_\varepsilon(t^{\frac{1}{\alpha}}) \|.$ Further, (2) and (5) yield
\[
\sup_{t < t_0} \left\| \frac{S_\varepsilon(t^{\frac{1}{\alpha}})}{t} N_\varepsilon(t^{\frac{1}{\alpha}}) \right\| \leq \sup_{t < t_0} \| S_\varepsilon(t^{\frac{1}{\alpha}}) \| \sup_{t < t_0} \| N_\varepsilon(t^{\frac{1}{\alpha}}) \|
\]
\[
= O_{\varepsilon \to 0}(\varepsilon^a),
\]
for some $t_0 > 0$ and $a \in \mathbb{R}.$ Also,
\[
\sup_{t < t_0} \left\| \frac{S_\varepsilon(t^{\frac{1}{\alpha}})N_\varepsilon(t^{\frac{1}{\alpha}})}{t} \right\| = O_{\varepsilon \to 0}(\varepsilon^a),
\]
for some $t_0 > 0$ and $a \in \mathbb{R}.$ Let now $\varepsilon \in (0,1)$ be fixed. We have
\[
\left\| \frac{S_\varepsilon(t^{\frac{1}{\alpha}})}{t} x - S_\varepsilon(0)H_\varepsilon x \right\| = \left\| \frac{S_\varepsilon(t^{\frac{1}{\alpha}})}{t} x - S_\varepsilon(t^{\frac{1}{\alpha}})H_\varepsilon x + S_\varepsilon(t^{\frac{1}{\alpha}})H_\varepsilon x - S_\varepsilon(0)H_\varepsilon x \right\|
\]
\[
\leq \left\| S_\varepsilon(t^{\frac{1}{\alpha}}) \right\| \left\| \frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t} x - S_\varepsilon(t^{\frac{1}{\alpha}})H_\varepsilon x \right\| + \left\| S_\varepsilon(t^{\frac{1}{\alpha}})H_\varepsilon x - S_\varepsilon(0)H_\varepsilon x \right\|
\]

By the first and the fifty properties in 5.5 as well as by the continuity of $t \to S_\varepsilon(t)(H_\varepsilon x)$ as zero, it follows that the last expression tend to zero as $t \to 0.$ Similarly, we have
\[ ||\frac{N_e(t^\frac{1}{\alpha})S_e(t^\frac{1}{\alpha})}{t}x - H_eS_e(0)x || = ||\frac{N_e(t^\frac{1}{\alpha})}{t}S_e(t^\frac{1}{\alpha})x - \frac{N_e(t^\frac{1}{\alpha})}{t}S_e(0)x + \frac{N_e(t^\frac{1}{\alpha})}{t}S_e(0)x - H_eS_e(0)x || \]

\[ \leq ||\frac{N_e(t^\frac{1}{\alpha})}{t}|| ||S_e(t^\frac{1}{\alpha})x - S_e(0)x || + ||\frac{N_e(t^\frac{1}{\alpha})}{t}(S_e(0)x) - H_e(S_e(0)x) || \]

Assumptions (4), (5) and (2) imply that the last expression tends to zero as \( t \to 0 \). Thus (5) is proved in both cases. \( \square \)

Now we define Colombeau type algebra as the factor algebra:

\[ SG(\mathbb{R}_+: L(X)) = SE_M(\mathbb{R}_+: L(X))/SN(\mathbb{R}_+: L(X)) \]

Elements of \( SG(\mathbb{R}_+: L(X)) \) will be denoted by \( S = [S_e] \), where \( (S_e)_e \) is a representative of the above class.

**Definition 5.6.** \( S \in SG(\mathbb{R}_+: L(X)) \) is a called a Colombeau \( C_0 \)-Semigroup if it has a representative \( (S_e)_e \) such that, for some \( \varepsilon_0 > 0 \), \( S_e \) is a \( C_0 \)-Semigroup, for every \( \varepsilon < \varepsilon_0 \).

**Example 5.1.** We take \( G = G(\mathbb{R}^+) \), and we define \( T_e(t)u_e(x) = u_e(x + \frac{t}{\alpha}) \).

Then \( T(t) = (T_e(t))_e \)

define a conformable semigroup on \( G \).

In the sequel we will use only representatives \( (S_e)_e \) of a Colombeau \( C_0 \)-semigroup \( S \) which are \( C_0 \)-semigroups, for \( \varepsilon \) small enough.

**Proposition 5.3.** Let \( (S_e)_e \) and \( (\tilde{S}_e)_e \) be representatives of a Colombeau \( C_0 \)-semigroup \( S \), with the infinitesimal generators \( A_e \), \( \varepsilon < \varepsilon_0 \), and \( \tilde{A}_e \), \( \varepsilon < \tilde{\varepsilon}_0 \), respectively, where \( \varepsilon_0 \) and \( \tilde{\varepsilon}_0 \) correspond (in the sense of definition 5.6 to \( (S_e)_e \) and \( (\tilde{S}_e)_e \), respectively.

Then, \( D(A_e) = D(\tilde{A}_e) \), for every \( \varepsilon < \varepsilon = \min \{ \varepsilon_0, \tilde{\varepsilon}_0 \} \) and \( A_e - \tilde{A}_e \) can be extended to an element of \( L(X) \), denoted again by \( A_e - \tilde{A}_e \).

Moreover, for every \( a \in \mathbb{R} \),

\[ ||A_e - \tilde{A}_e|| = O_{\varepsilon \rightarrow 0}(\varepsilon^a), \]  \hfill (5.6)

**Proof.** Denote \( N_e(S_e - \tilde{S}_e)_e \in SN(\mathbb{R}_+, L(X)) \).

Let \( \varepsilon < \varepsilon_0 \) be fixed and \( x \in X \). we have

\[ S_e(t^\frac{1}{\alpha})x - x = S_e(t^\frac{1}{\alpha})x - \tilde{S}_e(t^\frac{1}{\alpha})x + \tilde{S}_e(t^\frac{1}{\alpha})x - \tilde{S}_e(t^\frac{1}{\alpha})x = N_e(t^\frac{1}{\alpha})x \]
This implies by letting $t \to 0$, that $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$. Now we have

$$
(A_\varepsilon - (\tilde{A})_\varepsilon)x = \lim_{t \to 0} \frac{S_\varepsilon(t^{\frac{1}{\varepsilon}})x - x}{t} - \lim_{t \to 0} \frac{\tilde{S}_\varepsilon(t^{\frac{1}{\varepsilon}})x - x}{t} = \lim_{t \to 0} \frac{N_\varepsilon(t^{\tilde{\varepsilon}})}{t}x = H_\varepsilon x, \quad x \in D(A_\varepsilon)
$$

(5.7)

since $D(A_\varepsilon)$ is dense in $X$, properties (4), (5) and (7) imply that for every $a \in \mathbb{R}$,

$$
\|A_\varepsilon - \tilde{A}_\varepsilon\| = O_{\varepsilon \to 0}(\varepsilon^a).
$$

\[ \square \]

Now we define the infinitesimal generator of a Colombeau $C_0$-semigroup $S$.

Denote by $\mathcal{A}$ the set of pairs $((A_\varepsilon)_\varepsilon, (D(A_\varepsilon))_\varepsilon)$ where $A_\varepsilon$ is a closed linear operator on $X$ with the dense domain $D(A_\varepsilon) \subset X$, for every $\varepsilon \in (0,1)$. We introduce an equivalence relation in $\mathcal{A}$

$$
\left\{ (A_\varepsilon)_\varepsilon, (D(A_\varepsilon))_\varepsilon \right\} \sim \left\{ (\tilde{A}_\varepsilon)_\varepsilon, (D(\tilde{A}_\varepsilon))_\varepsilon \right\}
$$

if there exist $\varepsilon_0 \in (0,1)$ such that $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$, for every $\varepsilon < \varepsilon_0$, and for every $a \in \mathbb{R}$ ther exist $C > 0$ and $\varepsilon_a \leq \varepsilon_0$ such that, for $x \in A(A_\varepsilon), \| (A_\varepsilon - \tilde{A}_\varepsilon)x\| \leq C\varepsilon_a\|x\|, x \in D(A_\varepsilon), \varepsilon \leq \varepsilon_a$.

Since $A_\varepsilon$ has a dense domain in $X$, $R_\varepsilon := A_\varepsilon - \tilde{A}_\varepsilon$ ca, be extended to be an operator in $L_c(X)$ satisfying $\| (A_\varepsilon - \tilde{A}_\varepsilon)x\| = O_{\varepsilon \to 0}(\varepsilon^a)$, for every $a \in \mathbb{R}$. such an operator $R_\varepsilon$ is called the zero operator.

We denote by $\mathcal{A}$ the corresponding element of the quotient space $\mathcal{A}/ \sim$. Due to proposition 5.3, the following definition makes sense

**Definition 5.7.** $A \in \mathcal{A}/ \sim$ is the infinitesimal generator of a Colombeau $C_0$-semigroup $S$ if there exists a representative $(A_\varepsilon)_\varepsilon$ of $A$ such that $A_\varepsilon$ is the infinitesimal generator of $S_\varepsilon$, for $\varepsilon$ small enough.

By Pazy we have the following proposition

**Proposition 5.4.** Let $S$ be a Colombeau $C_0$-semigroup with the infinitesimal generator $A$. Then there exists $\varepsilon_0 \in (0,1)$ such that:

- Mapping $t \mapsto S_\varepsilon(t^{\frac{1}{\varepsilon}})x : \mathbb{R}_+ \to X$ is continuous for every $x \in X$ and $\varepsilon < \varepsilon_0$

- $\lim_{h \to 0} \int_t^{t+h} S_\varepsilon(s^{\frac{1}{\varepsilon}})xds_\alpha = S_\varepsilon(t^{\frac{1}{\varepsilon}})x, \quad \varepsilon < \varepsilon_0, \quad x \in X$

- $\int_0^t S_\varepsilon(s^{\frac{1}{\varepsilon}})xds_\alpha \in D(A_\varepsilon), \quad \varepsilon < \varepsilon_0, \quad x \in X$

- For every $x \in D(A_\varepsilon)$ and $t \geq 0$ $S_\varepsilon(t^{\frac{1}{\varepsilon}})x \in D(A_\varepsilon)$ and

$$
\frac{d^\alpha}{dt^\alpha} S_\varepsilon(t^{\frac{1}{\varepsilon}})x = A_\varepsilon S_\varepsilon(t^{\frac{1}{\varepsilon}})x = S_\varepsilon(t^{\frac{1}{\varepsilon}})A_\varepsilon x, \quad \varepsilon < \varepsilon_0
$$

(5.10)
Let \((S_e)_e\) and \((\tilde{S}_e)_\epsilon\) be representative of Colombeau \(C_0\)-semigroup \(S\), with infinitesimal generators \(A_e\) and \(\tilde{A}_e\), \(\epsilon < \epsilon_0\), respectively. Then, for every \(a \in \mathbb{R}\) and \(t \geq 0\)
\[
\| \frac{d^\alpha}{dt^\alpha} S_e(t^{\frac{1}{\alpha}}) - \tilde{A}_e S_e(t^{\frac{1}{\alpha}}) \| = O_{t \to 0}(\epsilon^a). \tag{5.11}
\]

For every \(x \in D(A_e)\) and every \(t, s \geq 0\),
\[
S_e(t^{\frac{1}{\alpha}})x - S_e(s^{\frac{1}{\alpha}})x = \int_s^t S_e(\tau^{\frac{1}{\alpha}})A_e x d\tau = \int_s^t A_e S_e(\tau^{\frac{1}{\alpha}})x d\tau. \tag{5.12}
\]

**Theorem 5.2.** Let \(S\) and \(\tilde{S}\) be Colombeau \(C_0\)-semigroups with infinitesimal generators \(A\) and \(\tilde{A}\), respectively. If \(A = \tilde{A}\) then \(S = \tilde{S}\).

**Proof.** Let \(\epsilon\) be small enough and \(x \in D(A_e) = D(\tilde{A}_e)\). Proposition 5.4 property 4 implies that for \(t \geq 0\), the mapping \(s \mapsto \tilde{S}_e((t-s)^{\frac{1}{\alpha}})S_e(s)^{\frac{1}{\alpha}}\), \(t \geq s \geq 0\) is differentiable and
\[
\frac{d^\alpha}{ds^\alpha} (\tilde{S}_e((t-s)^{\frac{1}{\alpha}})S_e(s^{\frac{1}{\alpha}})x) = -\tilde{A}_e \tilde{S}_e((t-s)^{\frac{1}{\alpha}})S_e(s^{\frac{1}{\alpha}})x
\]
\[
+ \tilde{S}_e((t-s)^{\frac{1}{\alpha}})A_e S_e(s^{\frac{1}{\alpha}})x, \quad t \geq s \geq 0.
\]
The assumption \(A = \tilde{A}\) implies that \(A_e = \tilde{A}_e + R_e\), where \(R_e\) is a zero operator. Since \(A_e\) commutes with \(\tilde{S}_e\), for every \(x \in D(A_e)\)
\[
\frac{d^\alpha}{ds^\alpha} (\tilde{S}_e((t-s)^{\frac{1}{\alpha}})S_e(s^{\frac{1}{\alpha}})x) = S_e((t-s)^{\frac{1}{\alpha}})R_e S_e(s^{\frac{1}{\alpha}})x, \quad t \geq s \geq 0.
\]
and this implies
\[
\tilde{S}_e((t-s)^{\frac{1}{\alpha}})S_e(s^{\frac{1}{\alpha}})x - \tilde{S}_e(t^{\frac{1}{\alpha}})x = \int_0^s \tilde{S}_e((t-u)^{\frac{1}{\alpha}})R_e S_e(u^{\frac{1}{\alpha}})x du, \quad t \geq s \geq 0. \tag{5.13}
\]

Putting \(s = t\) in (11), we obtain
\[
S_e(t^{\frac{1}{\alpha}}) - \tilde{S}_e(t^{\frac{1}{\alpha}})x = \int_0^t \tilde{S}_e((t-u)^{\frac{1}{\alpha}})R_e S_e(u^{\frac{1}{\alpha}})x du, \quad t \geq 0, \quad x \in D(A_e). \tag{5.14}
\]

Since \(D(A_e)\) is dense in \(X\), uniform boundedness of \(S\) and \(\tilde{S}\) on \([0, t]\) implies that (11) holds for every \(y \in X\). Let us prove that \((N_e)_e = (S_e - \tilde{S}_e)_e \in \mathcal{SC}(\mathbb{R}^+ : \mathcal{L}_e(X))\).
The formula (12) and definition 5.4 imply that for some \(C > 0\) and \(a, \tilde{a} \in \mathbb{R}\),
\[
\sup_{t \in [0, T]} \| N_e(t^{\frac{1}{\alpha}})x \| \leq \sup_{t \in [0, T]} \int_0^t \| \tilde{S}_e((t-u)^{\frac{1}{\alpha}}) \| \| R_e \| \| S_e(u^{\frac{1}{\alpha}}) \| \| x \| du \tag{5.15}
\]
\[
\leq TC e^{a+\tilde{a}} \| R_e \| \| x \|, \quad x \in X
\]

Since \(\| R_e \| = O_{t \to 0}(e^b)\), for every \(b \in \mathbb{R}\), \((N_e(t))_e\) satisfies condition (3) in definition 5.4. Condition (3) follows from the boundedness of \((\tilde{S}_e)_e\) on bounded domain \([0, t]\), the properties of \((R_e)_e\) and the following expression:
\[
\| \frac{N_e(t^{\frac{1}{\alpha}})}{t} \| \leq \| \int_0^t \tilde{S}_e((t-u)^{\frac{1}{\alpha}})R_e S_e(u^{\frac{1}{\alpha}})x du \| \leq \| \tilde{S}_e(t^{\frac{1}{\alpha}}) \| \| R_e \| \| S_e \| \leq \text{const}, \quad x \in X, \ t \leq t_0,
\]
for some $t_0 > 0$. Also,
\[
\lim_{t \to 0} \frac{N_\varepsilon(t^{\frac{1}{2}})}{t} = \lim_{t \to 0} \frac{S_\varepsilon(t^{\frac{1}{2}})x-x}{t} = \lim_{t \to 0} \frac{S_\varepsilon(t^{\frac{1}{2}})x-x}{t} = R \varepsilon, \quad \forall x \in D(A_\varepsilon).
\]
Since it is enough that (5) holds for a dense subset of $X$ see the remark after definition 5.5 this concludes the proof. \hfill \Box

6. MAIN RESULT

This section deals with applications of Colombeau $\alpha$-$C_0$-semigroups in solving a class of heat equations with singular potentials and singular data. We consider the problem
\[
\begin{cases}
\partial_t^\alpha u(t,x) = (\Delta - v(x))u(t,x) & \text{in } \mathbb{R}, \ t \in \mathbb{R}^+

u(0,t) = u_0(x) = \delta(x) \\
v(x) = \delta(x)
\end{cases}
\]
(6.1)
Before we discuss the problem we will defined some spaces on which we will work. First we set $\| \cdot \|_{L^2(\mathbb{R})} = \| \cdot \|_2$

**Definition 6.1.** We define $H^{2,\alpha}$ by:
\[
H^{2,\alpha} = \left\{ f \in L^2(\mathbb{R}), \quad \tilde{D}^\alpha f \in L^2(\mathbb{R}) \right\}
\]
with the norm
\[
\| f \|_{H^{2,\alpha}} = \sqrt{\| f \|_2^2 + \| \tilde{D}^\alpha f \|_2^2}
\]
The Colombeau algebra type defined as follows
\[
G_{H^{2,\alpha}} = E_M(H^{2,\alpha}) / \mathcal{N}(H^{2,\alpha})
\]
where $E_M(H^{2,\alpha})$ the vector space of nets $(G_\varepsilon)_\varepsilon \in H^{2,\alpha}$ with the property: for every $T > 0$ there exist $a \in \mathbb{R}$ such that
\[
\| G_\varepsilon \|_{H^{2,\alpha}} = O(\varepsilon^a)
\]
and $\mathcal{N}(H^{2,\alpha})$ the vector space of nets $(G_\varepsilon)_\varepsilon \in H^{2,\alpha}$ with the property: for every $T > 0$ for all $b \in \mathbb{R}$ such that
\[
\| G_\varepsilon \|_{H^{2,\alpha}} = O(\varepsilon^b)
\]
is a Colombeau type vector space.

**Definition 6.2.** $\mathcal{E}_{C^1,H^{2,\alpha}}([0,T],\mathbb{R})$ (respectively $\mathcal{N}_{C^1,H^{2,\alpha}}([0,T],\mathbb{R})$), $T > 0$, is the vector space of nets $(G_\varepsilon)_\varepsilon$ of functions
\[
G_\varepsilon \in C([0,T],H^{2,\alpha}) \cap C^1([0,T],L^2(\mathbb{R}))
\]
with the property: for every $T_1 \in (0,T)$ there exists $a \in \mathbb{R}$, (respectively, for every $a \in \mathbb{R}$) such that
\[
\max \left\{ \sup_{t \in [0,T]} \| G_\varepsilon \|_{H^{2,\alpha}}, \sup_{t \in [T_1,T]} \| \tilde{D}^\alpha G_\varepsilon \|_{L^2(\mathbb{R})} \right\} = O_{\varepsilon \to 0}(e^a)\]
The quotient space
\[ G_{C^1,H^{2,\alpha}}([0, T], \mathbb{R}) = \mathcal{E}_{C^1,H^{2,\alpha}}([0, T], \mathbb{R})/\mathcal{N}_{C^1,H^{2,\alpha}}([0, T], \mathbb{R}) \]
is a Colombeau type vector space.

**Remark 6.1.** The multiplication of potential \( V \in G_{H^{2,\alpha}} \) and \( u \in G_{C^1,H^{2,\alpha}}([0, T], \mathbb{R}) \) which is expected to be a solution to equation
\[
\begin{aligned}
\partial_t^\alpha u(t,x) &= (\Delta - \nu(x))u(t,x) \quad \text{in} \mathbb{R}, \; t \in \mathbb{R}^+ \\
u(0,t) &= u_0(x) = \delta(x) \\
\end{aligned}
\]
makes sense.

**Definition 6.3.** Let \( A \) be represented by a net \((A_\varepsilon)_{\varepsilon}\) of linear operators with the common domain \( H^{2,\alpha}(\mathbb{R}) \) and with ranges in \( L^2(\mathbb{R}) \). A generalized function \( G \in G_{C^1,H^{2,\alpha}} T > 0 \), is said to be a solution to equation \( \bar{D}^\alpha G = AG \) if
\[
\sup_{t \in [0,T]}\|\bar{D}^\alpha G_\varepsilon(t,\cdot) - A_\varepsilon G_\varepsilon(t,\cdot)\|_2 = O_{\varepsilon \to 0}(\varepsilon^a), \; \forall a \in \mathbb{R}.
\]

**Definition 6.4.** An element \( V \in G_{H^{2,\alpha}} \) is of logarithmic type if it has a representative \((V_\varepsilon)_{\varepsilon} \in \mathcal{E}_{C^1,H^{2,\alpha}}\) with the property
\[
\|V_\varepsilon\|_{H^{2,\alpha}} = O_{\varepsilon \to 0}(\ln \varepsilon^{-1})
\]
An element \( V \in G_{H^{2,\alpha}} \) is said to be of log-log type if it has a representative \((V_\varepsilon)_{\varepsilon} \in \mathcal{E}_{C^1,H^{2,\alpha}}\) with the property
\[
\|V_\varepsilon\|_{H^{2,\alpha}} = O_{\varepsilon \to 0}(\ln^a \ln \varepsilon^{-1})
\]
We set
\[
E(t,x) = \frac{1}{2\sqrt{\pi t}} \exp \frac{|x|^2}{4t}
\]

**Theorem 6.1.** Let \( V \in G_{H^{2,\alpha}} \) be of logarithmic type.

1. **Derivation of** \( A_\varepsilon u = (\Delta - V)u, \; u \in H^{2,\alpha}(\mathbb{R}) \), are infinitesimals generators of conformable semigroups \( T_\varepsilon \), for every \( \varepsilon > 0 \), and \((T_\varepsilon)_{\varepsilon}\) is a representative of a Colombeau conformable \( C_0 \)-semigroup.
\[
T(t) \in SG(\mathbb{R}^+, \mathcal{L}(L^2(\mathbb{R})))
\]
2. **Let** \( V \in G_{H^{2,\alpha}} \) and let \( T_\varepsilon \) be as in 1.
Then, for every \( T > 0 \), the problem 6.1 has unique solution in \( G_{H^{2,\alpha}}(\mathbb{R}) \).

**Proof.**

1. **Let** \( \varepsilon > 0 \) small enough. By the Feynman-Kac formula the operator \( A_\varepsilon \) is the infinitesimal generator of the corresponding semigroup
\[
T_\varepsilon(t)\psi(x) = \int_\Omega \left( \exp \left( - \int_0^t \alpha V_\varepsilon(\omega(s))ds \right) \psi(\omega(t^\alpha))d\mu_\varepsilon(\omega) \right)
\]
for $\psi \in L^2(\mathbb{R})$, where $\Omega = \prod_{t \geq 0} \mathbb{R}$ and $\mu_\epsilon$ is the Wiener measure concentrated at $x \in \mathbb{R}$.

Since $V$ is of logarithmic type, there exist $C > 0$ and $\eta \in (0, 1)$ such that

$$|T_\epsilon(t)\psi(x)| \leq \exp \left( t^{\alpha} \sup_{s \in \mathbb{R}} |V_\epsilon(s)| \right) \int_\Omega |\psi(\omega(t))| d\mu_\epsilon(\omega)$$

$$\leq \epsilon^{Ct^{\alpha}} \left( \frac{1}{2\sqrt{\pi t^{\alpha}}} \right) \int_{\mathbb{R}} \exp \left( -\frac{|x-y|^2}{4t^{\alpha}} \right) |\psi(y)| dy$$

Therefore, there exist $C_0 > 0$ such that

$$\sup_{t \in (0, T]} \|T_\epsilon(t)\psi(x)\|_2 \leq C_0 \epsilon^{CT^{\alpha}} \|\psi\|_2$$

Thus $(T_\epsilon)_\epsilon \in SG \left( \mathbb{R}^+, L(L^2(\mathbb{R})) \right)$

(2) (a) By the Duhamel principle, solution $u_\epsilon(t,x)$ to equation (6.1) satisfies

$$u_\epsilon(t,x) = \int_{\mathbb{R}} E(t^{\alpha}, x-y) b_\epsilon(y) dy$$

$$+ \int_0^t \int_{\mathbb{R}} E((t-s)^{\alpha}, x-y) V_\epsilon(y) u_\epsilon(s,y) dy ds$$

(6.2)

(6.3)

Young’s inequality implies

$$\|u_\epsilon(t,\cdot)\|_2 \leq \|b_\epsilon\|_2 + \int_0^t \|V_\epsilon(\cdot)\|_{L^\infty} \|u_\epsilon(s,\cdot)\|_2 ds$$

Gronwall’s inequality gives

$$\|u_\epsilon(t,\cdot)\|_2 \leq \|b_\epsilon\|_2 \exp \int_0^t \|V_\epsilon(\cdot)\|_{L^\infty}, \forall t \in (0, T]$$

Since $V \in GH^{2,\alpha}(\mathbb{R})$ is of logarithmic type and $(u_0)_\epsilon \in EH^{2,\alpha}(\mathbb{R})$, it follows that $\sup_{t \in (0,T]} \|u_\epsilon(t,\cdot)\|_2$ has a moderate bound. Differentiation of equation 6.2 with respect to some spatial variable $x$ gives

$$\frac{d}{dx} u_\epsilon(t,x) = \int_{\mathbb{R}} E(t^{\alpha}, y) \frac{d}{dx} b_\epsilon(x-y) dy$$

$$+ \int_0^t \int_{\mathbb{R}} E((t-s)^{\alpha}, y) \frac{d}{dx} V(x-y) u_\epsilon(s, x-y)$$

$$+ V(x-y) \frac{d}{dx} u_\epsilon(s, x-y) dy ds$$

then

$$\| \frac{d}{dx} u_\epsilon(t,\cdot) \|_2 \leq \| \frac{d}{dx} b_\epsilon \|_2 + \int_0^t \| \frac{d}{dx} V_\epsilon(\cdot) \|_{L^\infty} \|u_\epsilon(s,\cdot)\|_2 + \|V_\epsilon(\cdot)\|_{L^\infty} \| \frac{d}{dx} u_\epsilon(t,\cdot) \|_2$$

Also Gronwall’s inequality implies that $\sup_{t \in (0,T]} \|u_\epsilon(t,\cdot)\|_2$ is moderate. So $(u_\epsilon)_\epsilon \in EC^{1, H^{2,\alpha}}(\mathbb{R})$. 


(b) For uniqueness:
Let \( u_\varepsilon \) and \( v_\varepsilon \) two solution of (6.1).
\[ G_\varepsilon = u_\varepsilon - v_\varepsilon, \]
we get
\[ G_\varepsilon(t,x) = \int_{\mathbb{R}} E(t^\alpha, x-y)N_\varepsilon(y)dy \]
\[ + \int_0^t \int_{\mathbb{R}} E((t-s)^\alpha, x-y)V_\varepsilon(y)G_\varepsilon(s,y)dyds \]
\[ + \int_0^t \int_{\mathbb{R}} E((t-s)^\alpha, x-y)N_\varepsilon(y)dyds \]
where \( N_\varepsilon(x) = G(0,x) \), and \( N_\varepsilon = \frac{d^\alpha}{dt^\alpha} G_\varepsilon - (\Delta - V)G_\varepsilon \)
Then Young’s and Gronwall’s inequalities imply
\[ \|G_\varepsilon(t,.)\|_2 \leq \|N\|_2 + \int_0^t \|V_\varepsilon(s,.)\|_{L^\infty} \|G_\varepsilon(s,.)\|_2 ds + \int_0^t \|N_\varepsilon(s,.)\|_2 ds \alpha \]
thus \( G_\varepsilon \in \mathcal{N}(H^{2,\alpha}). \)

REFERENCES


[16] M. Stojanović, *Extension of Colombeau algebra to derivatives of arbitrary order* $D^\alpha$, $\alpha \in \mathbb{R}_+ \cup \{0\}$. Application to ODEs and PDEs with entire and fractional derivatives, Nonlinear Analysis 5 (2009), 5458–5475.

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CESARO-LIKE OPERATORS

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ABSTRACT. In previous work it was shown that the lower triangular generalized Hausdorff matrix $H_\alpha$ with nonzero entries $h_{nk} = (n + \alpha + 1)^{-1}$, for $\alpha \geq 0$, is subnormal on $\ell^2$ if and only if $\alpha = 0, 1, 2, \ldots$. For $0 < h \leq 1$, the weighted Cesaro operator $C'_h : \{a_n\} \to \{b_n\}$ on $\ell^2$, when $b_n = (a_0 + d_1a_1 + \cdots + d_na_n)/(n+1)d_n$, is subnormal when $d_j^2 = \Gamma(j+1)\Gamma(h)/\Gamma(j+h)$. In this paper we show that, when $d_j = \Gamma(j+1)\Gamma(h)/\Gamma(j+h)$, the square of the weights chosen above, then the corresponding operator $C_h$ is bounded on $\ell^2$ for $0 < h < 3/2$, that $H_\alpha$ is bounded on $\ell^2$ for all non-integer $\alpha < 0$, and that $C_h$ is closely related to $H_{h-1}$. This relationship leads to our main result that $C_h$ is only subnormal when $h = 1$, when it corresponds to the original Cesaro operator with $\alpha = 0$ and each $d_j = 1$.

1. INTRODUCTION AND SUMMARY OF RESULTS

The Cesaro operator $C : \{a_n\} \to \{b_n\}$ on $\ell^2$, where $b_n = (a_0 + a_1 + \cdots + a_n)/(n+1)$ was shown, in [6], to be subnormal, which answered a question raised in [1]. For $0 < h \leq 1$, the weighted Cesaro operator $C'_h$ on $\ell^2$, with

$$b_n = (a_0 + d_1a_1 + \cdots + d_na_n)/(n+1)d_n,$$

(1.1)

where $d_j^2 = \Gamma(j+1)\Gamma(h)/\Gamma(j+h)$, was shown to be subnormal in [5]. For $\alpha \geq 0$, another generalization of $C$, the lower triangular generalized Hausdorff matrix $H_\alpha$, with $h_{nk} = (n + \alpha + 1)^{-1}$, was shown, in [3], to be subnormal for $\alpha = 0, 1, 2, \ldots$. The question of the subnormality of $H_\alpha$ for noninteger $\alpha > 0$ was settled negatively in [7].

In this note we consider the transformation $C_h$ in [1] with weights

$$d_j = \frac{\Gamma(j+1)\Gamma(h)}{\Gamma(j+h)},$$

(1.2)

the square of the weights chosen in [5]. We show that these are also bounded operators on $\ell^2$ for $0 < h < 3/2$, and that they are closely related to the operators $H_\alpha$ in [3], but for $-1 < \alpha \leq 0$.

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We shall show that $H_\alpha$ is bounded for all non-integer $\alpha < 0$, but not subnormal. The relation between $C_h$ and $H_\alpha$ then yields our main result that $C_h$ is not subnormal, except for $h = 1$, when it is the original Cesaro operator studied in [1].

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2. A WEIGHTED CESARO OPERATOR.

For each sequence $\{d_n\}, d_n > 0$, define the transformation $C_d$ on $\ell^2$ by

$$C_d\{a_j\} = \{b_j\}, \quad \text{where} \quad b_j = \frac{(a_0d_0 + a_1d_1 + \cdots + a_jd_j)}{(j+1)d_j}. \quad (1.1')$$

If $0 < h \leq 1$ and $d_j = \Gamma(j+1)\Gamma(h)/\Gamma(j+h)$, it was shown in [5] that $C_d$ is a subnormal operator, a generalization of the result in [6] for the Cesaro operator. We now consider the case $d_j = \Gamma(j+1)\Gamma(h)/\Gamma(j+h)$, and denote the corresponding linear transformation on $\ell^2$ by $C_h$. Our goal is to determine whether or not $C_h$ is subnormal.

**Lemma 2.1.** If $0 < h < 3/2$ and $d_j = \Gamma(j+1)\Gamma(h)/\Gamma(j+h), j = 0, 1, 2, \ldots$, then $C_h$ is bounded on $\ell^2$. If $h \geq 3/2$, then $C_h$ is unbounded on $\ell^2$.

**Proof.** If $0 < h \leq 1$, the proof in [6] also applies here. Namely, if $h = 1$, then $C_h$ is the Cesaro operator and the result was proved in [1, p. 130]. Since $\{d_n\}$ is non-decreasing for $0 < h \leq 1$, the proof follows from the inequality

$$\left| \sum_{j=0}^{n} d_j a_j / d_n(n+1) \right| \leq \sum_{j=0}^{n} |a_j| / n+1.$$ 

Since $\|C\| = 2$, [1], $\|C_h\| \leq 2$ when $0 < h \leq 1$.

Now assume that $1 < h < 3/2$. The lower triangular matrix $C_h$ is a factorable matrix of the form $(C_h)_{nk} = a_n b_k$, where

$$a_n = \frac{1}{(n+1)d_n} = \frac{\Gamma(n+h)}{\Gamma(h)\Gamma(n+2)} = O(n^{h-2}),$$

and, by [8], page 47,

$$b_k = d_k = O(k^{1-h}).$$

Thus, for sufficiently large $n > N_0$ and $k > K_0$, we may assume that $C_h$ is a factorable matrix with entries $a'_n b'_k$, where $a'_n = n^{h-2}$ and $b'_k = k^{1-h}$. By Corollary 8(iii) on page 413 of [2], with $p = q = 2$, it follows that $C_h$ is a bounded operator on $\ell^2$ if and only if $2 - h > 1/2$ and $(2 - h) + (h - 1) \geq 1/2 + 1/2$.

Thus a necessary and sufficient condition for $C_h$ to be a bounded operator on $\ell^2$ is that $h < 3/2$. \qed
Lemma 2.2. The point spectrum of $C_h^*$ is the open disk
\[ \{ \lambda : |\lambda - \frac{1}{3 - 2h}| < \frac{1}{3 - 2h} \}, \]
for each $0 < h < 3/2$.

Proof. If $f = \{ f(n) \}$ is in $\ell^2$ and $C_h^* f = \lambda f$, then, as in the proof of Lemma 2.2 of [5], page 238, with $d_n = \Gamma(n+1)\Gamma(h)/\Gamma(n+h)$,
\[ f(n) = \frac{\Gamma(n+1)\Gamma(h)}{\Gamma(n+h)} \prod_{j=1}^n \left( 1 - \frac{\mu}{j} \right) f(0). \]
Suppose that
\[ |\lambda - \frac{1}{3 - 2h}|^2 < \left( \frac{1}{3 - 2h} \right)^2, \]
or, equivalently, if $\mu = 1/\lambda$, $2Re(\mu) > 3 - 2h$; i.e.,
\[ 2Re(\mu) = 3 - 2h + \epsilon \]
for some $\epsilon > 0$. Then
\[ |f(n)|^2 = \left| \frac{\Gamma(n+1)\Gamma(h)}{\Gamma(n+h)} \prod_{j=1}^n \left( 1 - \frac{\mu}{j} \right) f(0) \right|^2, \]
where
\[ \left| 1 - \frac{\mu}{j} \right|^2 = 1 - \frac{2Re(\mu)}{j} + \frac{|\mu|^2}{j^2} \]
\[ = 1 - \frac{3 - 2h + \epsilon}{j} + \frac{|\mu|^2}{j^2} \]
\[ \leq \exp \left( \frac{|\mu|^2}{j^2} - \frac{3 - 2h + \epsilon}{j} \right). \]

It follows from the estimate $\Gamma(n+1)\Gamma(h)/\Gamma(n+h) = O(n^{1-h})$, page 57 of [8], and the argument on page 130 of [1], that $f$ is in $\ell^2$, and hence every $\lambda$ satisfying $|\lambda - 1/(3 - 2h)| < 1/(3 - 2h)$ is an eigenvalue of $C_h^*$. That these are all of the eigenvalues follows as in [1].

\[ \square \]

Corollary 2.1. Let $T_h = I - C_h$, for $0 < h < 3/2$. The point spectrum of $T_h^*$ is
\[ \{ \lambda : |\lambda - (2 - 2h)/(3 - 2h)| < 1/(3 - 2h) \}. \]

Proof. $C_h^* f = \lambda f$ if and only if $(I - C_h^*) f = (1 - \lambda) f$. \[ \square \]

Following the constructions in [6], [5], and [3], for each $f$ in $\ell^2$, define $F$ by $F(z) = \langle f, \phi_z \rangle$, for all $|z - (2 - 2h)/(3 - 2h)| < 1/(3 - 2h)$, where $T_h^* \phi_z = z \phi_z$, and $\phi_z(0) = 1$. Let $\mathcal{H}$ denote the set of all functions $F$, and define $\|F\|_{\mathcal{H}} = \|f\|_{\ell^2}$.

Theorem 2.1. For all $1/2 \leq h \leq 3/2$, $\mathcal{H}$ is the space of functions $F(z)$ analytic for $|z - (2 - 2h)/(3 - 2h)| < 1/(3 - 2h)$. The operator $T_h$ in $\ell^2$ is unitarily equivalent to the operator in $\mathcal{H}$ which maps $F(z)$ into $zF(z)$ in $\mathcal{H}$. The functions $\psi_0(z) =$
$1, \psi_n(z) = d_n(z-1)^{-n}z(z-1/2)\cdots(z-(n-1)/n), n \geq 1,$ form an orthonormal basis for $\mathcal{H}$. The reproducing kernel function for $\mathcal{H}$ is

$$K(w,z) = \sum_{n} \psi_n(z) \overline{\psi_n(w)} = 3F_2\left(\begin{array}{c} -\frac{w}{1-w} - h + 1, -\frac{z}{1-z} - h + 1 \end{array}; h, h, 1 \right).$$

**Proof.** The correspondence $f \leftrightarrow F$ is 1-1 since the functions $\phi_n$ span a dense subset of $\ell^2$ (or $H^2$) when $1/2 \leq h < 3/2$. The numbers $\lambda_n = 1/n, n = 1, 2, \ldots,$ are in the point spectrum of $T_h^*$, and $\phi_n$ is a polynomial of degree $n - 1$, by the proof of Lemma 2.2. So, the span of all of the $\phi_n$ includes all of the polynomials, which are dense in $\ell^2$.

The proofs of the other statements in Theorem 1 are the same as in those on page 216 of [8] and page 238 of [5]. An orthonormal basis for $\mathcal{H}$ can also be derived by a direct computation.

Note that, for $0 < h \leq 1/2, 0$ is not in the spectrum of $T_h$, so that the above proof that $f \leftrightarrow F$ is a 1-1 correspondence does not hold. This observation is due to Stefan Maurer, as is the following lemma.

**Lemma 2.3.** For $0 < h < 1/2$, the correspondence $f \leftrightarrow F$ between $\ell^2$ and $\mathcal{H}$ is not 1-1.

**Proof.** Since $\Gamma(n+h)/\Gamma(n+1) = O(n^{h-1})$ from page 47 of [8], $f = \{f(n+1) / \Gamma(n+1)\}$ is in $\ell^2$ when $0 < h < 1/2$. We shall show that the corresponding analytic function $F(z)$ in $\mathcal{H}$ vanishes for all real $z$, and hence vanishes identically.

From the proof of Lemma 2.2, $F(z) = \langle f, \phi \rangle$, where $\phi(0) = 1$ and

$$\phi_n(z) = \frac{\Gamma(n+1)\Gamma(h)}{\Gamma(n+h)} \prod_{j=1}^{n} \left(1 - \frac{1}{jz^2}\right), \quad n = 1, 2, \ldots.$$

So, for real $c = 1/z, c = 3/2 - h + \varepsilon/2 > 1 + \varepsilon/2$, and

$$F(z) = 1 + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \left(1 - \frac{1}{jz^2}\right) = 1 + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \left(1 - \frac{c}{j}\right) = 0,$$

using the binomial expansion for $(1 + x)^{c-1}$ with $x = -1$.

A consequence of Lemma 2.3 is that $T_h$ in $\ell^2$ is not unitarily equivalent to $F(z) \rightarrow zF(z)$ in $\mathcal{H}$ for $0 < h < 1/2$. For our purposes it will be sufficient to continue with an analysis of $T_h$ for $1/2 \leq h < 3/2$.

The mapping $z \rightarrow z/(1-z)$ takes the disk $|z-(2-2h)/(3-2h)| < 1/(3-2h)$ onto the half plane $\text{Re}(w) > -h + 1/2$. The inverse map is $w \rightarrow w/(1+w)$.

Let $\mathcal{K}$ denote the set of functions $F$ of the form $F(z) = G(z/(1+z))$ for some $G$ in $\mathcal{H}$, where $\text{Re}(z) > -h + 1/2$. Using the conformal mapping, the orthonormal basis $\{(-1)^n \psi_n(z)\}$ for $\mathcal{H}$ is mapped into the orthonormal basis $\psi_n(z) = 1, \psi_n(z) = (d_n/n!)z(z-1)\cdots(z-n+1), n > 1,$ for $\mathcal{K}$, and the reproducing kernel function for $\mathcal{K}$ is

$$K(w,z) = \sum_{n} \psi_n(z) \overline{\psi_n(w)} = 3F_2\left(\begin{array}{c} w+h-1, z+h-1 \end{array}; h, h, 1 \right).$$
It follows, as on page 218 of [6] and page 239 of [5] that, for \(0 < a, b < 1\), \(a\) and \(b\) belong to \(\mathcal{K}\) and
\[
\langle a, b \rangle_{\mathcal{K}} = \sum_{n=0}^{\infty} \frac{(1-a)^n(1-b)^n}{d_n^2} = \sum_{n=0}^{\infty} (1-a)^n(1-b)^n \frac{\Gamma^2(n+1)\Gamma^2(h)}{\Gamma^2(n+h)}
= F(h, h; 1; (1-a)(1-b)).
\]

Let \(\mathcal{K}'\) denote all functions \(g(z)\) of the form \(g(z) = f(z + h - 1)\) for some \(f\) in \(\mathcal{K}\), and define
\[
\|g(z)\|_{\mathcal{K}'} = \|f(z)\|_{\mathcal{K}}.
\]

Then, for \(0 < a, b < 1\), \(a\) and \(b\) belong to \(\mathcal{K}'\), and
\[
\langle a, b \rangle_{\mathcal{K}'} = (ab)^{h-1} \langle a, b \rangle_{\mathcal{K}} = (ab)^{h-1} F(h, h; 1; (1-a)(1-b)).
\]

This formula also appears on page 262 of [3], where it was shown to hold for \(h \geq 1\). In our case, \(0 < h \leq 1\), by Lemma 2.1. However, a careful review of the results on page 262 of [3] shows that they hold not only for \(h \geq 1\), but also for \(0 < h < 1\); i.e., \(-1 < \alpha < 0\). The result that the operator \(H_\alpha\) in [3] is bounded for \(-1 < \alpha < 0\) will be proved in the next section. It follows, from a comparison of the Hilbert space \(\mathcal{K}'\) above with the Hilbert space \(\mathcal{K}_\alpha\) in [3], with \(\alpha = h - 1\) and \(0 < h < 1\), that they have the same orthonormal basis. Hence they are identical.

By Theorem 1, there is a unitary operator \(U_1\), from \(\ell^2\) onto \(\mathcal{K}\) such that \(C_h\) is unitarily equivalent to \(F(z) \to F(z)/(1 + z)\) in \(\mathcal{K}\), and a unitary operator \(U_2\), from \(\ell^2\) onto \(\mathcal{K}'\), such that \(C_h\) is unitarily equivalent to \(F(z) \to F(z)/(z + 2 - h)\) in \(\mathcal{K}'\). Since \(\mathcal{K}_{h-1}\) in [3] is the same space as \(\mathcal{K}'\), the operator \(H_{h-1}\) in [3] is unitarily equivalent to \(F(z) \to F(z)/(1 + z)\) in \(\mathcal{K}'\). Therefore we have the following identities relating \(C_h\) and \(H_{h-1}\):
\[
H_{h-1}[I - (1-h)C_h] = C_h, \quad (2.1)
\]
and
\[
C_h[I - (h-1)H_{h-1}] = H_{h-1}. \quad (2.2)
\]

For certain values of \(h\) these identities can be simplified. If \(1/2 < h \leq 1\), then, by the proof of Lemma 2.1, \(\|C_h\| \leq 2\) and \(\|(1-h)C_h\| \leq 2(1-h) < 1\). Thus \(I - (1-h)C_h\) is invertible and
\[
H_{h-1} = C_h[I - (1-h)C_h]^{-1}. \quad (2.1')
\]

It now follows, from [4] that, for \(1/2 < h \leq 1\),
\[
C_h = H_{h-1}[I - (h-1)H_{h-1}]^{-1}, \quad (2.2')
\]
that \(I - (1-h)C_h\) and \(I - (h-1)H_{h-1}\) are bounded inverses of each other, and that \(C_hH_{h-1} = H_{h-1}C_h\).

Since \(C_h\) is bounded for \(0 < h < 3/2\) and, as we shall show in Lemma 3.1, \(H_\alpha\) is bounded for \(\alpha > -1\); i.e., \(h > 0\), the operators in (2.1') and (2.2') are bounded for \(0 < h < 3/2\). Since all of the matrix entries are polynomials in \(h\) (or rational functions of \(H\) with poles at the negative integers), and (2.1') and (2.2') hold for
$1/2 < h \leq 1$, it follows that they are also true for $0 < h < 3/2$. (Thus, as pointed out by Larry Zalcman, we need not appeal to the Identity Theorem for analytic functions.)

In the next section we shall use identities (2.1') and (2.2') to show that $C_h$ is not subnormal, except for $h = 1$.

### 3. Generalized Hausdorff Operators $H_\alpha$ for $\alpha < 0$.

As in [3], let $H_\alpha$ denote the lower triangular matrix with entries $h_{nk} = 1/(n + \alpha + 1)$. From [4], $H_\alpha$ is bounded on $\ell^2$ for $\alpha \geq 0$.

**Lemma 3.1.** For $-1 < \alpha < 0$, $H_\alpha$ is bounded on $\ell^2$.

**Proof.** Note that the nonzero terms of $H_\alpha$ are

$$h_{nk} = \frac{d_n}{n+1}, \quad \text{where} \quad d_n = \frac{(n+1)}{(n+\alpha+1)}.$$

Therefore $H_\alpha = DC$, where $D$ is the diagonal matrix with diagonal entries $(d_n)$ and $C$ is the Cesaro matrix of order 1.

Since $C$ is known to be a bounded operator on $\ell^2$ ([1]), to prove the lemma it will be sufficient to show that the sequence $\{d_n\}$ is bounded.

By inspection, for each $n \geq 0$,

$$|d_n| \leq \max \left\{ \frac{1}{\varepsilon}, 2 \right\},$$

where $\varepsilon = \alpha + 1$ denotes the distance from $\alpha$ to $-1$. Therefore $D$ is bounded by $1/\varepsilon$ and $H_\alpha$ is bounded on $\ell^2$ for $-1 < \alpha < 0$. □

For completeness we present a more general result. The symbol $\mathbb{N}$ denotes the set of positive integers.

**Lemma 3.2.** For any $k \in \mathbb{N}$, if $-k < \alpha < -k + 1$, then $H_\alpha$ is bounded on $\ell^2$.

**Proof.** As in the proof of Lemma 3.1, define $\varepsilon_1 = \alpha + k$, the distance from $\alpha$ to $-k$, and $\varepsilon_2 = \alpha + k - 1$, the distance from $\alpha$ to $-k + 1$. Then $D$ is bounded by $\max\{k/\varepsilon_1, (k+1)/\varepsilon_2\}$, and $H_\alpha$ is bounded on $\ell^2$ for $-k < \alpha < -k + 1$. □

**Remark.** It is clear from the above arguments that $H_\alpha$ is also bounded on $\ell^p$ for $p > 1$, and for all non-integer $\alpha < 0$.

The next lemma is needed to show that $H_\alpha$ is not subnormal on $\ell^2$ for non-integer $\alpha < 0$.

**Lemma 3.3.** If $H_\alpha$ is subnormal on $\ell^2$ for any $\alpha > -1$, and $n \in \mathbb{N}$, then $H_{\alpha+n}$ is also subnormal on $\ell^2$.

**Proof.** Since the proof is by induction, it is sufficient to provide a proof for $n = 1$. If the first row and column of $H_\alpha$ are deleted, the resulting matrix is $H_{\alpha+1}$. Thus, $H_{\alpha+1}$ may be regarded as the restriction of $H_\alpha$ to the closed invariant subspace of $\ell^2$ consisting of all sequences $\{a_n\} \in \ell^2$ of the form $\{0, a_1, a_2, \ldots\}$. Since the
restriction of a subnormal operator to a closed invariant subspace is clearly also subnormal, \( H_{\alpha+1} \) is subnormal.

**Corollary 3.1.** \( H_\alpha \) is not subnormal for \(-1 < \alpha < 0\).

*Proof.* The proof is by contradiction. Let \(-1 < \alpha < 0\) and assume that \( H_\alpha \) is subnormal. Then \( 0 < \alpha + 1 < 1 \), and, by Lemma 3.3, \( H_{\alpha+1} \) is subnormal. But this contradicts the result in [7] that \( H_\alpha \) is not subnormal for any non-integer \( \alpha > 0 \).

**Corollary 3.2.** \( H_\alpha \) is not subnormal for any non-integer \( \alpha < 0 \).

*Proof.* The result clearly follows from Corollary 2, Lemma 3.3, and an induction argument.

**Theorem 3.1.** \( C_h \) is not subnormal for any \( 0 < h < 3/2 \), except for \( h = 1 \), the Cesaro operator.

*Proof.* If \( h \neq 1 \), and \( C_h \) is subnormal, then so is \( |I - (1-h)C_h|^{-1} \). Thus, by (2.1'), so is \( H_{h-1} \). If \( 0 < h < 1 \), we have a contradiction to Corollary 2. If \( 1 < h < 3/2 \), we have a contradiction to the result in [7] that \( H_\alpha \) is not subnormal for non-integer \( \alpha > 0 \).

**References**


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FG-COUPLED FIXED POINT THEOREMS FOR VARIOUS CONTRACTIONS IN PARTIALLY ORDERED METRIC SPACES

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ABSTRACT. In this paper we introduce FG-coupled fixed point, which is a generalization of coupled fixed point for nonlinear mappings in partially ordered complete metric spaces. We discuss existence and uniqueness theorems of FG-coupled fixed points for different contractive mappings. Our theorems generalizes the results of Gnana Bhaskar and Lakshmikantham [1].

1. INTRODUCTION

Fixed point theory has many applications in nonlinear analysis. In [3–5] the authors presented fixed point theorems in partially ordered metric spaces and their applications. As a generalization of fixed points, in [2] Guo and Lakshmikantham introduced the concept of abstract coupled fixed points for some operators, thereafter Gnana Bhaskar and Lakshmikantham in [1] introduced coupled fixed points and mixed monotone property for contractive mappings on partially ordered metric spaces. They proved interesting coupled fixed point results in [1]. An interesting application of their result is that it can be used to find the solution of periodic boundary value problem, moreover it guarantees the uniqueness of the solution. Followed by this, several authors established new coupled fixed point theorems in partially ordered complete metric spaces and in cone metric spaces. In [6] Sabetghadam, Masiha and Sanatpour proved generalization of results of Gnana Bhaskar and Lakshmikantham in cone metric spaces.

In this paper we introduce a new concept which is a generalization of coupled fixed point and prove existence theorems for contractive mappings in partially ordered metric spaces. Some examples are also discussed to illustrate our results. We recall the basic definitions.

Definition 1.1 ([1]). Let \((X, \leq)\) be a partially ordered set and \(F : X \times X \to X\). We say that \(F\) has the mixed monotone property if \(F(x, y)\) is monotone non decreasing
in x and is monotone non increasing in y, that is for any \( x, y \in X \)

\[
x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y) \quad \text{and} \quad y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).
\]

**Definition 1.2** ([1]). We call an element \((x, y) \in X \times X\) a coupled fixed point of the mapping \(F\) if \(F(x, y) = x, \quad F(y, x) = y\).

## 2. Main Results

**Definition 2.1.** Let \((X, \leq_{P_1})\) and \((Y, \leq_{P_2})\) be two partially ordered sets and \(F : X \times Y \to X\) and \(G : Y \times X \to Y\) be two mappings. An element \((x, y) \in X \times Y\) is said to be an FG-coupled fixed point if

\[
F(x, y) = x \quad \text{and} \quad G(y, x) = y.
\]

**Note 2.1.** If \(X = Y\) and \(F = G\) then FG-coupled fixed point becomes coupled fixed point. An element \((x, y) \in X \times Y\) is FG-coupled fixed point if and only if \((y, x) \in Y \times X\) is GF-coupled fixed point.

**Note 2.2.** Let \((X, d_X, \leq_{P_1})\) and \((Y, d_Y, \leq_{P_2})\) be two partially ordered metric spaces, then we define the partial order \(\leq\) and metric \(d\) on \(X \times Y\) as follows:

For all \((x, y), (u, v) \in X \times Y\),

\[
(x, y) \leq (u, v) \iff x \leq_{P_1} u \quad \text{and} \quad y \geq_{P_2} v \quad \text{and} \quad d((x, y), (u, v)) = d_X(x, u) + d_Y(y, v).
\]

**Definition 2.2.** Let \((X, \leq_{P_1})\) and \((Y, \leq_{P_2})\) be two partially ordered sets and \(F : X \times Y \to X\) and \(G : Y \times X \to Y\). We say that \(F\) and \(G\) have mixed monotone property if \(F\) and \(G\) are monotone increasing in first variable and monotone decreasing in second variable, i.e., if for all \((x, y) \in X \times Y\),

\[
x_1, x_2 \in X, x_1 \leq_{P_1} x_2 \Rightarrow F(x_1, y) \leq_{P_1} F(x_2, y) \quad \text{and} \quad G(y, x_1) \geq_{P_2} G(y, x_2) \quad \text{and} \quad y_1, y_2 \in Y, y_1 \leq_{P_2} y_2 \Rightarrow F(x, y_1) \geq_{P_2} F(x, y_2) \quad \text{and} \quad G(y_1, x) \leq_{P_2} G(y_2, x).
\]

**Note 2.3.** Let \(F : X \times Y \to X\) and \(G : Y \times X \to Y\) be two mappings, then for \(n \geq 1\),

\[
F^n(x, y) = F(F^{n-1}(x, y), G^{n-1}(y, x)) \quad \text{and} \quad G^n(y, x) = G(G^{n-1}(y, x), F^{n-1}(x, y))
\]

where for all \(x \in X\) and \(y \in Y\), \(F^0(x, y) = x\) and \(G^0(y, x) = y\).

**Theorem 2.1.** Let \((X, d_X, \leq_{P_1})\) and \((Y, d_Y, \leq_{P_2})\) be two partially ordered complete metric spaces and \(F : X \times Y \to X\) and \(G : Y \times X \to Y\) be two continuous functions having the mixed monotone property. Assume that there exist \(k, l \in [0, 1)\) with

\[
d_X(F(x, y), F(u, v)) \leq \frac{k}{2}[d_X(x, u) + d_Y(y, v)], \forall x \geq_{P_1} u, y \leq_{P_2} v; \tag{2.1}
\]

\[
d_Y(G(y, x), G(v, u)) \leq \frac{l}{2}[d_Y(y, v) + d_X(x, u)], \forall x \leq_{P_1} u, y \geq_{P_2} v. \tag{2.2}
\]
If there exist \((x_0,y_0) \in X \times Y\) such that \(x_0 \leq F(x_0,y_0)\) and \(y_0 \geq G(y_0,x_0)\), then there exist \((x,y) \in X \times Y\) such that \(x = F(x,y)\) and \(y = G(y,x)\).

**Proof.** By hypothesis there exists \((x_0,y_0) \in X \times Y\) such that 
\[
x_0 \leq F(x_0,y_0) = x_1 \text{ (say)} \quad \text{and} \quad y_0 \geq G(y_0,x_0) = y_1 \text{ (say)}.
\]

For \(n = 1, 2, 3, \ldots\) define \(x_{n+1} = F(x_n,y_n)\) and \(y_{n+1} = G(y_n,x_n)\) then we get 
\[
x_{n+1} = F^{n+1}(x_0,y_0) \quad \text{and} \quad y_{n+1} = G^{n+1}(y_0,x_0).
\]

Then we can easily prove that \(\{x_n\}\) is an increasing sequence in \(X\) and \(\{y_n\}\) is a decreasing sequence in \(Y\) by using the mixed monotone property of \(F\) and \(G\).

**Claim:** For \(n \in \mathbb{N}\)
\[
d_X(F^{n+1}(x_0,y_0),F^n(x_0,y_0)) \leq \frac{k}{2} \left(\frac{k+l}{2}\right)^{n-1} \left[d_X(x_1,x_0)+d_Y(y_1,y_0)\right], \quad (2.3)
\]
\[
d_Y(G^{n+1}(y_0,x_0),G^n(y_0,x_0)) \leq \frac{l}{2} \left(\frac{k+l}{2}\right)^{n-1} \left[d_Y(y_1,y_0)+d_X(x_1,x_0)\right]. \quad (2.4)
\]

We will use the fact that \(\{x_n\}\) is an increasing sequence in \(X\) and \(\{y_n\}\) is a decreasing sequence in \(Y\), (2.1), (2.2) and symmetric property of \(d_Y\) to prove the claim.

For \(n = 1\),
\[
d_X(F^2(x_0,y_0),F(x_0,y_0)) = d_X(F(F(x_0,y_0),G(y_0,x_0)),F(x_0,y_0)) \\
\leq \frac{k}{2} \left[d_X(F(x_0,y_0),x_0)+d_Y(G(y_0,x_0),y_0)\right] \\
= \frac{k}{2} \left[d_X(x_1,x_0)+d_Y(y_1,y_0)\right].
\]

Similarly
\[
d_Y(G^2(y_0,x_0),G(y_0,x_0)) \leq \frac{l}{2} \left[d_Y(y_0,y_1)+d_X(x_0,x_1)\right].
\]

Now assume the claim for \(n \leq m\) and check for \(n = m + 1\).

Consider,
\[
d_X(F^{m+2}(x_0,y_0),F^{m+1}(x_0,y_0)) \\
= d_X(F(F^{m+1}(x_0,y_0),G^{m+1}(y_0,x_0)),F(F^m(x_0,y_0),G^m(y_0,x_0))) \\
\leq \frac{k}{2} \left[d_X(F^{m+1}(x_0,y_0),F^m(x_0,y_0))+d_Y(G^{m+1}(y_0,x_0),G^m(y_0,x_0))\right] \\
\leq \frac{k}{2} \left(\frac{k+l}{2}\right)^{m-1} \left[d_X(x_1,x_0)+d_Y(y_1,y_0)\right] + \frac{l}{2} \left(\frac{k+l}{2}\right)^{m-1} \left[d_Y(y_1,y_0)+d_X(x_1,x_0)\right] \\
= \frac{k}{2} \left(\frac{k+l}{2}\right)^m \left[d_X(x_1,x_0)+d_Y(y_1,y_0)\right].
\]

Similarly we can show that
\[
d_Y(G^{m+2}(y_0,x_0),G^{m+1}(y_0,x_0)) \leq \frac{l}{2} \left(\frac{k+l}{2}\right)^m \left[d_X(x_1,x_0)+d_Y(y_1,y_0)\right].
\]
Thus the claim is true for all $n \in \mathbb{N}$. Using the result obtained we prove that \{x_n\} is a Cauchy sequence in $X$ and \{y_n\} is a Cauchy sequence in $Y$.

For $m \geq n$ consider,

$$d_X(F^m(x_0,y_0), F^n(x_0,y_0))$$

\[\leq d_X(F^m(x_0,y_0), F^{m-1}(x_0,y_0)) + d_X(F^{m-1}(x_0,y_0), F^{m-2}(x_0,y_0)) + \cdots + d_X(F(x_0,y_0), F(x_0,y_0))\]

\[\leq \frac{k}{2} \left( \frac{k+1}{2} \right)^{m-2} [d_X(x_1,x_0) + d_Y(y_1,y_0)] + \frac{k}{2} \left( \frac{k+1}{2} \right)^{m-3} [d_X(x_1,x_0) + d_Y(y_1,y_0)] + \cdots + \frac{k}{2} \frac{(k+1)^{n-1}}{(2^{n-1})} [d_X(x_1,x_0) + d_Y(y_1,y_0)]\]

\[\leq \frac{k}{2} \left( \frac{\theta^{n-1}}{1-\theta} \right) [d_X(x_1,x_0) + d_Y(y_1,y_0)]; \text{ where } \theta = \frac{k+1}{2} < 1 \text{ as } n \to \infty.\]

That is \{F^n(x_0,y_0)\} \to x_0 is a Cauchy sequence in $(X,d_X)$.

Similarly we get \{G^n(y_0,x_0)\} \to y_0 is a Cauchy sequence in $(Y,d_Y)$.

Since $(X,d_X)$ and $(Y,d_Y)$ are complete metric spaces, we have

$$\lim_{n \to \infty} F^n(x_0,y_0) = x \text{ and } \lim_{n \to \infty} G^n(y_0,x_0) = y \text{ for some } (x,y) \in X \times Y.$$

Now we can prove that $(x,y)$ is an FG-coupled fixed point by using the continuity of $F$ and $G$. For that consider,

$$d_X(F(x,y), x) = \lim_{n \to \infty} d_X(F^n(x_0,y_0), G^n(y_0,x_0), F^n(x_0,y_0))$$

$$= \lim_{n \to \infty} d_X(F^{n+1}(x_0,y_0), F^n(x_0,y_0)) = 0.$$

That is $F(x,y) = x$.

In a similar manner we can prove that $G(y,x) = y$.

This completes the proof. \qed

**Example 2.1.** Let $X = (-\infty, 0]$ and $Y = [0, \infty)$ with usual order and usual metric. Define $F : X \times Y \to X$ and $G : Y \times X \to Y$ as $F(x,y) = \frac{x-y}{3}$ and $G(y,x) = \frac{y-x}{3}$, then it is easy to check the conditions (2.1) and (2.2) for $F$ and $G$ with $k = \frac{2}{3}, l = \frac{2}{3}$. Here $(0,0)$ is the unique FG-coupled fixed point.

We obtain the result of Gnana Bhaskar and Lakshmikantham [1] as a corollary of our result.

**Corollary 2.1.** [1, Theorem 2.1] Let $(X, \leq)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X,d)$ is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on $X$. Assume that there exist $k \in [0,1)$ with

$$d(F(x,y), F(u,v)) \leq \frac{k}{2} [d(x,u) + d(y,v)], \forall x \geq u, y \leq v.$$
If there exists \( x_0, y_0 \in X \) such that \( x_0 \leq F(x_0, y_0) \) and \( y_0 \geq F(y_0, x_0) \), then there exist \( x, y \in X \) such that \( x = F(x, y) \) and \( y = F(y, x) \).

**Proof.** Take \( X = Y, F = G \) and \( k = l \) in Theorem 2.1, we get the result. \( \square \)

**Remark 2.1.** By adding to the hypothesis of Theorem 2.1 the condition: for every \((x,y), (x_1, y_1) \in X \times Y\) there exists a \((u,v) \in X \times Y\) that is comparable to both \((x,y)\) and \((x_1, y_1)\), we can obtain a unique FG-coupled fixed point.

In the following theorem, we prove the uniqueness of FG-coupled fixed point using the above condition.

**Theorem 2.2.** Let \((X, d_X, \leq_{P_1})\) and \((Y, d_Y, \leq_{P_2})\) be two partially ordered complete metric spaces and \(F : X \times Y \to X\) and \(G : Y \times X \to Y\) be two continuous functions having the mixed monotone property. Assume that for every \((x,y), (x_1, y_1) \in X \times Y\) there exists a \((u,v) \in X \times Y\) that is comparable to both \((x,y)\) and \((x_1, y_1)\) and there exist \(k, l \in [0, 1)\) with

\[
d_X(F(x,y), F(u,v)) \leq \frac{k}{2}[d_X(x,u) + d_Y(y,v)], \forall x \geq_{P_1} u, y \leq_{P_2} v, \tag{2.1}\]

\[
d_Y(G(y,x), G(u,v)) \leq \frac{l}{2}[d_Y(y,v) + d_X(x,u)], \forall y \leq_{P_2} u, x \geq_{P_1} v. \tag{2.2}\]

If there exist \((x_0, y_0) \in X \times Y\) such that \(x_0 \leq_{P_1} F(x_0, y_0)\) and \(y_0 \geq_{P_2} G(y_0, x_0)\), then there exist unique \((x,y) \in X \times Y\) such that \(x = F(x,y)\) and \(y = G(y,x)\).

**Proof.** Following as in Theorem 2.1 we obtain the existence of FG-coupled fixed point. Now we show the uniqueness part.

Suppose that \((x^*, y^*) \in X \times Y\) is another FG-coupled fixed point, then we show that \(d((x,y), (x^*, y^*)) = 0\), where \(x = \lim_{n \to \infty} F^n(x_0, y_0)\) and \(y = \lim_{n \to \infty} G^n(y_0, x_0)\).

Claim: For any two points \((x_1, y_1), (x_2, y_2) \in X \times Y\) which are comparable,

\[
d_X(F^n(x_1, y_1), F^n(x_2, y_2)) \leq \left(\frac{k+1}{2}\right)^n [d_X(x_1, x_2) + d_Y(y_1, y_2)], \tag{2.5}\]

\[
d_Y(G^n(y_1, x_1), G^n(y_2, x_2)) \leq \left(\frac{k+1}{2}\right)^n [d_Y(y_1, y_2) + d_X(x_1, x_2)]. \tag{2.6}\]

Without loss of generality assume that \((x_2, y_2) \leq_{P_1} (x_1, y_1)\).

We will use (2.1), (2.2) and symmetric property of \(d_Y\) to prove the claim.

For \(n = 1\) consider,

\[
d_X(F(x_1, y_1), F(x_2, y_2)) \leq \frac{k}{2}[d_X(x_1, x_2) + d_Y(y_1, y_2)] \leq \frac{k+1}{2}[d_X(x_1, x_2) + d_Y(y_1, y_2)],
\]

\[
d_Y(G(y_1, x_1), G(y_2, x_2)) \leq \frac{l}{2}[d_Y(y_1, y_2) + d_X(x_1, x_2)] \leq \frac{k+1}{2}[d_Y(y_1, y_2) + d_X(x_1, x_2)].
\]
That is our claim is true for \( n = 1 \).

Assume that it is true for \( n \leq m \) and check for \( n = m + 1 \).

Consider,

\[
d_X(F^{m+1}(x_1, y_1), F^{m+1}(x_2, y_2)) = d_X(F(F^m(x_1, y_1), G^m(y_1, x_1)), F(F^m(x_2, y_2), G^m(y_2, x_2)))
\]

\[
\leq \frac{k}{2} [d_X(F^m(x_1, y_1), F^m(x_2, y_2)) + d_Y(G^m(y_1, x_1), G^m(y_2, x_2))]
\]

\[
\leq \frac{k}{2} \left( \frac{k + l}{2} \right)^m [d_X(x_1, x_2) + d_Y(y_1, y_2)] + \left( \frac{k + l}{2} \right)^m [d_X(x_1, x_2) + d_Y(y_1, y_2)]
\]

\[
\leq \left( \frac{k + l}{2} \right)^{m+1} [d_X(x_1, x_2) + d_Y(y_1, y_2)].
\]

Similarly,

\[
d_Y(G^{m+1}(y_1, x_1), G^{m+1}(y_2, x_2)) \leq \left( \frac{k + l}{2} \right)^{m+1} [d_Y(y_1, y_2) + d_X(x_1, x_2)].
\]

Thus our claim is true for all \( n \in \mathbb{N} \).

To prove the uniqueness we consider two cases:

Case 1: Assume \((x, y)\) is comparable to \((x^*, y^*)\) with respect to the ordering in \(X \times Y\). We have,

\[
d((x, y), (x^*, y^*)) = d_X(x, x^*) + d_Y(y, y^*)
\]

\[
= d_X(F^n(x, y), F^n(x^*, y^*)) + d_Y(G^n(y, x), G^n(y^*, x^*))
\]

\[
\leq \left( \frac{k + l}{2} \right)^n [d_X(x, x^*) + d_Y(y, y^*)] + \left( \frac{k + l}{2} \right)^n [d_Y(y, y^*) + d_X(x, x^*)]
\]

\[
= 2 \left( \frac{k + l}{2} \right)^n [d_X(x, x^*) + d_Y(y, y^*)] \to 0 \text{ as } n \to \infty.
\]

This implies that \((x, y) = (x^*, y^*)\).

Case 2: If \((x, y)\) is not comparable to \((x^*, y^*)\), then by the hypothesis there exist \((u, v) \in X \times Y\) that is comparable to both \((x, y)\) and \((x^*, y^*)\), which implies that \((v, u) \in Y \times X\) is comparable to both \((y, x)\) and \((y^*, x^*)\).

Consider

\[
d((x, y), (x^*, y^*)) = d\left((F^n(x, y), G^n(y, x)), (F^n(x^*, y^*), G^n(y^*, x^*))\right)
\]

\[
\leq d\left((F^n(x, y), G^n(y, x)), (F^n(u, v), G^n(v, u))\right)
\]

\[
+ d\left((F^n(x^*, y^*), G^n(y^*, x^*)), (F^n(u, v), G^n(v, u))\right)
\]

\[
= d_X(F^n(x, y), F^n(u, v)) + d_Y(G^n(y, x), G^n(v, u))
\]

\[
+ d_X(F^n(x^*, y^*), F^n(u, v)) + d_Y(G^n(y^*, x^*), G^n(v, u))
\]

\[
\leq \left( \frac{k + l}{2} \right)^n [d_X(x, u) + d_Y(y, v)] + \left( \frac{k + l}{2} \right)^n [d_Y(y, v) + d_X(x, u)]
\]
Let suppose that for all $[1, \text{Theorem 2.4}]$ parable to both $\text{Proof.}$

Take $\varphi$ FG-coupled.

Following the proof of Theorem 2.1 we only have to show that there exist $(y, x)$ in $X$ and $y$ in $Y$, such that $x = F(x, y)$ and $y = G(y, x)$.

Therefore we have $x = F(x, y)$.

We can prove that $G(y, x) = y$. This completes the proof.

**Corollary 2.2.** [1, Theorem 2.4] In addition to the hypothesis of corollary 2.1, suppose that for all $(x, y), (z, t) \in X \times X$ there exists a $(u, v) \in X \times X$ that is comparable to both $(x, y)$ and $(z, t)$, then $F$ has a unique coupled fixed point.

**Proof.** Take $X = Y, F = G$ and $k = l$ in Theorem 2.1, we get the result. 

**Theorem 2.3.** Let $(X, d_X, \leq_{P_1})$ and $(Y, d_Y, \leq_{P_2})$ be two partially ordered complete metric spaces. Assume that $X$ and $Y$ have the following properties:

(i) if a non decreasing sequence $\{x_n\} \rightarrow x$ in $X$, then $x_n \leq_{P_1} x$ for all $n$

(ii) if a non increasing sequence $\{y_n\} \rightarrow y$ in $Y$, then $y_n \geq_{P_2} y$ for all $n$.

Let $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$ be two functions having the mixed monotone property. Assume that there exist $k, l \in [0, 1)$ with

\[
\begin{align*}
d_X(F(x, y), F(u, v)) & \leq \frac{k}{2} [d_X(x, u) + d_Y(y, v)], \forall x \geq_{P_1} u, y \leq_{P_2} v, \\
d_Y(G(y, x), G(v, u)) & \leq \frac{l}{2} [d_Y(y, v) + d_X(x, u)], \forall x \leq_{P_1} u, y \geq_{P_2} v.
\end{align*}
\]

If there exist $(x_0, y_0) \in X \times Y$ such that $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$, then there exist $(x, y) \in X \times Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

**Proof.** Following the proof of Theorem 2.1 we only have to show that $(x, y)$ is an FG-coupled fixed point. Recall from the proof of Theorem 2.1 that $\{x_n\}$ is increasing in $X$ and $\{y_n\}$ is decreasing in $Y$, $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$. We have,

\[
\begin{align*}
d_X(F(x, y), x) & \leq d_X(F(x, y), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \\
& = d_X(F(x, y), F(F^n(x_0, y_0), G^n(y_0, x_0))) + d_X(F^{n+1}(x_0, y_0), x).
\end{align*}
\]

By (i) and (ii), $x \geq_{P_1} F^n(x_0, y_0)$ and $y \leq_{P_2} G^n(y_0, x_0)$, therefore by (2.1)

\[
\begin{align*}
d_X(F(x, y), x) & \leq \frac{k}{2} [d_X(x, F^n(x_0, y_0)) + d_Y(y, G^n(y_0, x_0))] + d_X(F^{n+1}(x_0, y_0), x) \\
& \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{align*}
\]

Therefore we have $F(x, y) = x$.

Similarly we can prove that $G(y, x) = y$. This completes the proof.
We obtain the result of Gnana Bhaskar and Lakshmikantham [1] as a corollary of our result.

**Corollary 2.3.** [1, Theorem 2.2] Let \((X, \leq)\) be a partially ordered set and suppose there is a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. Assume that \(X\) has the following property:

(i) if a non decreasing sequence \(\{x_n\} \to x\), then \(x_n \leq x\) for all \(n\)

(ii) if a non increasing sequence \(\{y_n\} \to y\), then \(y_n \geq y\) for all \(n\).

Let \(F : X \times X \to X\) be a mapping having the mixed monotone property on \(X\). Assume that there exist non negative \(k, l\) with

\[
\text{Claim: For } n \in \mathbb{N}
\]

\[
d_X(F(x,y), F(u,v)) \leq k[d(x,u) + d(y,v)], \forall x \geq u, y \leq v.
\]

If there exist \(x_0, y_0 \in X\) such that \(x_0 \leq F(x_0, y_0)\) and \(y_0 \geq F(y_0, x_0)\), then there exist \(x, y \in X\) such that \(x = F(x, y)\) and \(y = F(y, x)\).

**Proof.** Take \(X = Y\), \(F = G\) and \(k = l\) in Theorem 2.3, we get the result.

**Remark 2.2.** By adding to the hypothesis of Theorem 2.3 the condition: for every \((x, y), (x_1, y_1) \in X \times Y\) there exists a \((u, v) \in X \times Y\) that is comparable to both \((x, y)\) and \((x_1, y_1)\), we can obtain a unique FG-coupled fixed point.

**Theorem 2.4.** Let \((X, d_X, \leq_{p_1})\) and \((Y, d_Y, \leq_{p_2})\) be two partially ordered complete metric spaces and \(F : X \times Y \to X\) and \(G : Y \times X \to Y\) be two continuous functions having the mixed monotone property. Assume that there exist non negative \(k, l\) with \(k + l < 1\) such that

\[
d_X(F(x,y), F(u,v)) \leq kd_X(x,u) + ld_Y(y,v); \forall x \geq_{p_1} u, y \leq_{p_2} v, \quad (2.7)
\]

\[
d_Y(G(y,x), G(v,u)) \leq kd_Y(y,v) + ld_X(x,u); \forall x \leq_{p_1} u, y \geq_{p_2} v. \quad (2.8)
\]

If there exist \((x_0, y_0) \in X \times Y\) such that \(x_0 \leq_{p_1} F(x_0, y_0)\) and \(y_0 \geq_{p_2} G(y_0, x_0)\), then there exist \((x, y) \in X \times Y\) such that \(x = F(x, y)\) and \(y = G(y, x)\).

**Proof.** Following as in Theorem 2.1 we get an increasing sequence \(\{x_n\}\) in \(X\) and a decreasing sequence \(\{y_n\}\) in \(Y\) where \(x_{n+1} = F(x_n, y_n) = F^{n+1}(x_0, y_0)\) and \(y_{n+1} = G(y_n, x_n) = G^{n+1}(y_0, x_0)\).

**Claim:** For \(n \in \mathbb{N}\)

\[
d_X(F^{n+1}(x_0, y_0), F^n(x_0, y_0)) \leq (k + l)^n[d_X(x_1, x_0) + d_Y(y_1, y_0)], \quad (2.9)
\]

\[
d_Y(G^{n+1}(y_0, x_0), G^n(y_0, x_0)) \leq (k + l)^n[d_Y(y_1, y_0) + d_X(x_1, x_0)]. \quad (2.10)
\]

By using (2.7), (2.8) and symmetric property of \(d_Y\) we prove the claim.

For \(n = 1\) consider,

\[
d_X(F^2(x_0, y_0), F(x_0, y_0)) = d_X(F(F(x_0, y_0), G(y_0, x_0)), F(x_0, y_0))
\]

\[
\leq kd_X(F(x_0, y_0), x_0) + ld_Y(G(y_0, x_0), y_0)
\]
\[ kdx(x_1,x_0) + ldy(y_1,y_0) \leq (k+l)[d_x(x_1,x_0) + d_y(y_1,y_0)].\]

Similarly, \( dy(G^2(y_0,x_0), G(y_0,x_0)) \leq (k+l)[d_y(y_1,y_0) + d_x(x_1,x_0)]. \)  

Assume the result is true for \( n \leq m, \) then check for \( n = m+1. \)

Consider,  
\[
d_x(F^{m+2}(x_0,y_0), F^{m+1}(x_0,y_0)) \\
= d_x(F(F^{m+1}(x_0,y_0), G^{m+1}(y_0,x_0)), F(F(x_0,y_0), G^m(y_0,x_0))) \\
\leq kdx(F^{m+1}(x_0,y_0), F^m(x_0,y_0)) + ldy(G^{m+1}(y_0,x_0), G^m(y_0,x_0)) \\
\leq k(k+l)^m[d_x(x_1,x_0) + d_y(y_1,y_0)] + l(k+l)^m[d_y(y_1,y_0) + d_x(x_1,x_0)] \\
\leq (k+l)^{m+1}[d_x(x_1,x_0) + d_y(y_1,y_0)].
\]

Similarly we can prove that 
\[
d_y(G^{m+2}(y_0,x_0), G^{m+1}(y_0,x_0)) \leq (k+l)^{m+1} [d_y(y_1,y_0) + d_x(x_1,x_0)].
\]

Thus the claim is true for all \( n \in \mathbb{N}. \)

Next we prove that \( \{x_n\} \) is a Cauchy sequence in \( X \) and \( \{y_n\} \) is a Cauchy sequence in \( Y \) using (2.9) and (2.10) respectively.

For \( m \geq n \) consider, 
\[
d_x(F^m(x_0,y_0), F^n(x_0,y_0)) \\
\leq d_x(F^m(x_0,y_0), F^{m-1}(x_0,y_0)) + d_x(F^{m-1}(x_0,y_0), F^{m-2}(x_0,y_0)) \\
+ \cdots + d_x(F^{n+1}(x_0,y_0), F^n(x_0,y_0)) \\
\leq (k+l)^{m-1}[d_x(x_1,x_0) + d_y(y_1,y_0)] + (k+l)^{m-2}[d_x(x_1,x_0) + d_y(y_1,y_0)] \\
+ \cdots + (k+l)^n[d_x(x_1,x_0) + d_y(y_1,y_0)] \\
= \{(k+l)^{m-1} + (k+l)^{m-2} + \cdots + (k+l)^n\}[d_x(x_1,x_0) + d_y(y_1,y_0)] \\
\leq \frac{\delta^n}{1-\delta}[d_x(x_1,x_0) + d_y(y_1,y_0)] \to 0 \text{ as } n \to \infty; \text{ where } \delta = k+l < 1.
\]

This implies that \( \{F^n(x_0,y_0)\} \) is a Cauchy sequence in \( X. \) One can show that \( \{G^n(y_0,x_0)\} \) is a Cauchy sequence in \( Y. \) Since \( (X,d_x) \) and \( (Y,d_y) \) are complete metric spaces we have \( (x,y) \in X \times Y \) such that \( \lim_{n \to \infty} F^n(x_0,y_0) = x \) and \( \lim_{n \to \infty} G^n(x_0,y_0) = y. \) In the same lines as in Theorem 2.1 we can show that \( (x,y) \in X \times Y \) is an FG-coupled fixed point. Hence the proof.  \( \square \)

**Example 2.2.** Let \( X = (-\infty,0] \) and \( Y = [0,\infty) \) with usual order and usual metric. Define \( F : X \times Y \to X \) and \( G : Y \times X \to Y \) as \( F(x,y) = \frac{4x-3y}{17} \) and \( G(y,x) = \frac{4y-3x}{17}, \) then it is easy to check that \( F \) and \( G \) satisfies the conditions (2.7) and (2.8) for \( k = \frac{4}{17}, l = \frac{3}{17}. \) Here \( (0,0) \) is the unique FG-coupled fixed point.
Remark 2.3. By adding to the hypothesis of Theorem 2.4 the condition: for every $(x, y), (x_1, y_1) \in X \times Y$ there exists a $(u, v) \in X \times Y$ that is comparable to both $(x, y)$ and $(x_1, y_1)$, we can obtain a unique FG-coupled fixed point.

In the following theorem we obtain uniqueness of FG-coupled fixed point using the above condition.

**Theorem 2.5.** Let $(X, d_X, \leq P_1)$ and $(Y, d_Y, \leq P_2)$ be two partially ordered complete metric spaces and $F : X \times Y \to X$ and $G : Y \times X \to Y$ be two continuous functions having the mixed monotone property. Assume that for every $(x, y), (x_1, y_1) \in X \times Y$ there exists a $(u, v) \in X \times Y$ that is comparable to both $(x, y)$ and $(x_1, y_1)$ and there exist non negative $k, l$ with $k + l < 1$ such that

\[
d_X(F(x, y), F(u, v)) \leq kd_X(x, u) + ld_Y(y, v); \forall x \geq P_1 u, y \leq P_2 v, \tag{2.7}
\]

\[
d_Y(G(y, x), G(v, u)) \leq kd_Y(y, v) + ld_X(x, u); \forall x \leq P_1 u, y \geq P_2 v. \tag{2.8}
\]

If there exist $(x_0, y_0) \in X \times Y$ such that $x_0 \leq P_1 F(x_0, y_0)$ and $y_0 \geq P_2 G(y_0, x_0)$, then there exist unique $(x, y) \in X \times Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

**Proof.** Following as in Theorem 2.4 we obtain existence of FG-coupled fixed point. Now we prove the uniqueness part.

Suppose that $(x^*, y^*) \in X \times Y$ is another FG-coupled fixed point, then we show that $d((x, y), (x^*, y^*)) = 0$, where $x = \lim_{n \to \infty} F^n(x_0, y_0)$ and $y = \lim_{n \to \infty} G^n(y_0, x_0)$.

Claim: For any two points $(x_1, y_1), (x_2, y_2) \in X \times Y$ which are comparable,

\[
d_X(F^n(x_1, y_1), F^n(x_2, y_2)) \leq (k + l)^n[d_X(x_1, x_2) + d_Y(y_1, y_2)], \tag{2.11}
\]

\[
d_Y(G^n(y_1, x_1), G^n(y_2, x_2)) \leq (k + l)^n[d_Y(y_1, y_2) + d_X(x_1, x_2)]. \tag{2.12}
\]

Without loss of generality assume that $(x_2, y_2) \leq (x_1, y_1)$. Using (2.7) and (2.8) we prove the claim.

For $n = 1$ we have,

\[
d_X(F(x_1, y_1), F(x_2, y_2)) \leq kd_X(x_1, x_2) + ld_Y(y_1, y_2)
\]

\[
\leq (k + l)[d_X(x_1, x_2) + d_Y(y_1, y_2)].
\]

Now assume that the result is true for $n \leq m$ and check for $n = m + 1$.

Consider,

\[
d_X(F^{m+1}(x_1, y_1), F^{m+1}(x_2, y_2))
\]

\[
= d_X(F(F^m(x_1, y_1), G^m(y_1, x_1)), F(F^m(x_2, y_2), G^m(y_2, x_2)))
\]

\[
\leq kd_X(F^m(x_1, y_1), F^m(x_2, y_2)) + ld_Y(G^m(y_1, x_1), G^m(y_2, x_2))
\]

\[
\leq k(k + l)^m[d_X(x_1, x_2) + d_Y(y_1, y_2)] + l(k + l)^m[d_Y(y_1, y_2) + d_X(x_1, x_2)]
\]

\[
= (k + l)^m[d_X(x_1, x_2) + d_Y(y_1, y_2)].
\]

Similarly we get,

\[
d_Y(G^{m+1}(y_1, x_1), G^{m+1}(y_2, x_2)) \leq (k + l)^{m+1}[d_Y(y_1, y_2) + d_X(x_1, x_2)].
\]
Thus the claim is true for all $n \in \mathbb{N}$.

To prove the uniqueness we use the inequalities (2.11) and (2.12). We consider two cases:

Case 1: Assume $(x,y)$ is comparable to $(x^*,y^*)$ with respect to the ordering in $X \times Y$. Now consider,

$$d((x,y),(x^*,y^*)) = d_X(x,x^*) + d_Y(y,y^*)$$

$$= d_X(F^n(x,y),F^n(x^*,y^*)) + d_Y(G^n(y,x),G^n(y^*,x^*))$$

$$\leq (k+l)^n[d_X(x,x^*) + d_Y(y,y^*)] + (k+l)^n[d_Y(y,y^*) + d_X(x,x^*)]$$

$$= 2(k+l)^n[d_X(x,x^*) + d_Y(y,y^*)] \to 0 \text{ as } n \to \infty.$$

This implies that $(x,y) = (x^*,y^*)$.

Case 2: If $(x,y)$ is not comparable to $(x^*,y^*)$, then by the hypothesis there exist $(u,v) \in X \times Y$ that is comparable to both $(x,y)$ and $(x^*,y^*)$. Now consider

$$d((x,y),(x^*,y^*))$$

$$= d((F^n(x,y),G^n(y,x)),(F^n(x^*,y^*),G^n(y^*,x^*)))$$

$$\leq d((F^n(x,y),G^n(y,x)),(F^n(u,v),G^n(v,u)))$$

$$+ d((F^n(u,v),G^n(v,u)),(F^n(x^*,y^*),G^n(y^*,x^*)))$$

$$\leq (k+l)^n[d_X(x,u) + d_Y(y,v)] + (k+l)^n[d_Y(y,v) + d_X(x,u)]$$

$$+ (k+l)^n[d_X(x^*,u) + d_Y(y^*,v)] + (k+l)^n[d_Y(y^*,v) + d_X(x^*,u)]$$

$$= 2(k+l)^n[d_X(x,u) + d_Y(y,v)] + 2(k+l)^n[d_X(x^*,u) + d_Y(y^*,v)] \to 0 \text{ as } n \to \infty.$$

which implies $(x,y) = (x^*,y^*)$.

Hence the uniqueness of FG-coupled fixed point is proved. \hfill \Box

The above result is valid for any two mappings $F$ and $G$ if the spaces satisfies a condition as shown in the following theorem.

**Theorem 2.6.** Let $(X,d_X,\leq_{P_1})$ and $(Y,d_Y,\leq_{P_2})$ be two partially ordered complete metric spaces. Assume that $X$ and $Y$ have the following properties:

\begin{enumerate}[(i)]
  \item if a non decreasing sequence $\{x_n\} \to x$ in $X$, then $x_n \leq_{P_1} x$ for all $n$
  \item if a non increasing sequence $\{y_n\} \to y$ in $Y$, then $y_n \geq_{P_2} y$ for all $n$.
\end{enumerate}

Let $F : X \times Y \to X$ and $G : Y \times X \to Y$ be two functions having the mixed monotone property. Assume that there exist non negative $k$, $l$ with $k+l < 1$

\begin{equation}
  d_X(F(x,y),F(u,v)) \leq kd_X(x,u) + ld_Y(y,v), \forall x \geq_{P_1} u, y \leq_{P_2} v,
\end{equation}

\begin{equation}
  d_Y(G(y,x),G(v,u)) \leq kd_Y(y,v) + ld_X(x,u), \forall x \leq_{P_1} u, y \geq_{P_2} v.
\end{equation}

If there exist $(x_0,y_0) \in X \times Y$ such that $x_0 \leq_{P_1} F(x_0,y_0)$ and $y_0 \geq_{P_2} G(y_0,x_0)$, then there exist $(x,y) \in X \times Y$ such that $x = F(x,y)$ and $y = G(y,x)$. 

Following the proof of Theorem 2.4 we only have to show that \((x, y)\) is an FG-coupled fixed point. Recall from the proof of Theorem 2.4 that \(\{x_n\}\) is increasing in \(X\) and \(\{y_n\}\) is decreasing in \(Y\), \(\lim_{n \to \infty} F^n(x_0, y_0) = x\) and \(\lim_{n \to \infty} G^n(y_0, x_0) = y\).

We have
\[
d_X(F(x, y), x) \leq d_X(F(x, y), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x)
= d_X(F(x, y), F(F^n(x_0, y_0), G^n(y_0, x_0))) + d_X(F^{n+1}(x_0, y_0), x).
\]

By (i) and (ii) we have \(x \geq \rho_1 F^n(x_0, y_0)\) and \(y \leq \rho_2 G^n(y_0, x_0)\). Therefore using (2.7) we get
\[
d_X(F(x, y), x) \leq kd_X(x, F^n(x_0, y_0)) + ld_Y(y, G^n(y_0, x_0)) + d_X(F^{n+1}(x_0, y_0), x) \to 0 \text{ as } n \to \infty.
\]
That is \(F(x, y) = x\).

Similarly we get \(G(y, x) = y\).

This completes the proof.

\[\square\]

**Remark 2.4.** By adding to the hypothesis of Theorem 2.6 the condition: for every \((x, y), (x_1, y_1) \in X \times Y\) there exists a \((u, v) \in X \times Y\) that is comparable to both \((x, y)\) and \((x_1, y_1)\), we can obtain a unique FG-coupled fixed point.

**Remark 2.5.** By putting \(k = l = \frac{k}{2}\) in theorems 2.4, 2.5, 2.6 we get theorems 2.1, 2.4, 2.2 of Gnana Bhaskar and Lakshmikantham [1] respectively.

**Theorem 2.7.** Let \((X, d_X, \leq \rho_1)\) and \((Y, d_Y, \leq \rho_2)\) be two partially ordered complete metric spaces and \(F : X \times Y \to X\) and \(G : Y \times X \to Y\) be two continuous functions having the mixed monotone property. Assume that there exist non negative \(k, l\) with \(k + l < 1\) such that
\[
d_X(F(x, y), F(u, v)) \leq kd_X(x, F(x, y)) + ld_X(u, F(u, v)), \forall x \geq \rho_1 u, y \leq \rho_2 v, \quad (2.13)
\]
\[
d_Y(G(y, x), G(v, u)) \leq kd_Y(y, G(y, x)) + ld_Y(v, G(v, u)), \forall x \leq \rho_1 u, y \geq \rho_2 v. \quad (2.14)
\]
If there exist \((x_0, y_0) \in X \times Y\) such that \(x_0 \leq \rho_1 F(x_0, y_0)\) and \(y_0 \geq \rho_2 G(y_0, x_0)\), then there exist \((x, y) \in X \times Y\) such that \(x = F(x, y)\) and \(y = G(y, x)\).

**Proof.** By using the mixed monotone property of \(F\) and \(G\) and given conditions on \(x_0\) and \(y_0\) it is easy to show that \(\{x_n\}\) is an increasing sequence in \(X\) and \(\{y_n\}\) is a decreasing sequence in \(Y\) where \(x_{n+1} = F(x_n, y_n) = F^{n+1}(x_0, y_0)\) and \(y_{n+1} = G(y_n, x_n) = G^{n+1}(y_0, x_0)\).

**Claim:** For \(n \in \mathbb{N}\)
\[
d_X(F^{n+1}(x_0, y_0), F^n(x_0, y_0)) \leq \left(\frac{l}{1-k}\right)^n d_X(x_1, x_0), \quad (2.15)
\]
\[
d_Y(G^{n+1}(y_0, x_0), G^n(y_0, x_0)) \leq \left(\frac{k}{1-l}\right)^n d_Y(y_1, y_0). \quad (2.16)
\]
Using the contraction on \(F\) and \(G\) and symmetric property on \(d_Y\) we prove the claim.
For \( n = 1 \) consider,
\[
d_X(F^2(x_0,y_0), F(x_0,y_0)) = d_X(F(F(x_0,y_0), G(y_0,x_0)), F(x_0,y_0))
\leq kd_X(F(x_0,y_0), F^2(x_0,y_0)) + ld_X(x_0,F(x_0,y_0))
\]
i.e, \((1-k)d_X(F^2(x_0,y_0), F(x_0,y_0)) \leq ld_X(x_0,F(x_0,y_0)) = ld_X(x_0,x_1)\)
\[
i.e, d_X(F^2(x_0,y_0), F(x_0,y_0)) \leq \left( \frac{l}{1-k} \right) d_X(x_0,x_1).
\]

Hence for \( n = 1 \), the claim is true. Now assume the claim for \( n \leq m \) and check for \( n = m + 1 \).

Consider,
\[
d_X(F^{m+2}(x_0,y_0), F^{m+1}(x_0,y_0))
\]
\[
= d_X(F(F^{m+1}(x_0,y_0), G^{m+1}(y_0,x_0)), F(F^m(x_0,y_0), G^m(y_0,x_0)))
\leq kd_X(F^{m+1}(x_0,y_0), F^{m+2}(x_0,y_0)) + ld_X(F^m(x_0,y_0), F^{m+1}(x_0,y_0))
\]
i.e, \((1-k)d_X(F^{m+2}(x_0,y_0), F^{m+1}(x_0,y_0)) \leq ld_X(F^m(x_0,y_0), F^{m+1}(x_0,y_0))\)
\[
\leq l\left( \frac{l}{1-k} \right)^m d_X(x_0,x_1)
\]
i.e, \(d_X(F^{m+2}(x_0,y_0), F^{m+1}(x_0,y_0)) \leq \left( \frac{l}{1-k} \right)^{m+1} d_X(x_0,x_1)\).

Similarly we get,
\[
d_Y(G^{m+2}(y_0,x_0), G^{m+1}(y_0,x_0)) \leq \left( \frac{k}{1-l} \right)^{m+1} d_Y(y_0,y_1).
\]

Thus our claim is true for all \( n \in \mathbb{N} \). Next we prove that \( \{F^n(x_0,y_0)\} \) and \( \{G^n(y_0,x_0)\} \) are Cauchy sequences in \( X \) and \( Y \) using (2.15) and (2.16) respectively.

For \( m > n \) consider,
\[
d_X(F^m(x_0,y_0), F^n(x_0,y_0))
\leq d_X(F^m(x_0,y_0), F^{m-1}(x_0,y_0)) + d_X(F^{m-1}(x_0,y_0), F^{m-2}(x_0,y_0))
\]
\[
+ \cdots + d_X(F^{n+1}(x_0,y_0), F^n(x_0,y_0))
\leq \left( \frac{l}{1-k} \right)^{m-1} d_X(x_0,x_1) + \left( \frac{l}{1-k} \right)^{m-2} d_X(x_0,x_1) + \cdots + \left( \frac{l}{1-k} \right)^n d_X(x_0,x_1)
\]
\[
= \left\{ \left( \frac{l}{1-k} \right)^{m-1} + \left( \frac{l}{1-k} \right)^{m-2} + \cdots + \left( \frac{l}{1-k} \right)^n \right\} d_X(x_0,x_1)
\]
\[
\leq \left( \frac{\delta^n}{1-\delta} \right) d_X(x_0,x_1) \to 0 \text{ as } n \to \infty; \text{ where } \delta = \frac{l}{1-k} < 1.
\]

This implies that \( \{F^n(x_0,y_0)\} \) is a Cauchy sequence in \( X \). In a similar manner we can prove that \( \{G^n(y_0,x_0)\} \) is a Cauchy sequence in \( Y \).
Since \((X, d_X)\) and \((Y, d_Y)\) are complete metric spaces we have \((x, y) \in X \times Y\) such that \(\lim_{n \to \infty} F^n(x_0, y_0) = x\) and \(\lim_{n \to \infty} G^n(y_0, x_0) = y\). Proceeding as in Theorem 2.1 we get \((x, y)\) is an FG-coupled fixed point. This completes the proof. \(\square\)

**Example 2.3.** Let \(X = [1, 2]\) and \(Y = [-2, -1]\) with usual metric and usual order. Define \(F : X \times Y \to X\) and \(G : Y \times X \to Y\) by \(F(x, y) = \frac{x}{4} + 1\) and \(G(y, x) = \frac{y}{4} - 1\), then we can see that the conditions (2.13) and (2.14) for \(F\) and \(G\) are satisfied with \(k = \frac{1}{3}, l = \frac{1}{2}\). Here \((\frac{1}{3}, \frac{1}{2})\) is the FG-coupled fixed point.

**Theorem 2.8.** Let \((X, d_X, \leq_{P_1})\) and \((Y, d_Y, \leq_{P_2})\) be two partially ordered complete metric spaces. Assume that \(X\) and \(Y\) have the following properties:

(i) if a non decreasing sequence \(\{x_n\} \to x\) in \(X\), then \(x_n \leq_{P_1} x\) for all \(n\)

(ii) if a non increasing sequence \(\{y_n\} \to y\) in \(Y\), then \(y_n \geq_{P_2} y\) for all \(n\).

Let \(F : X \times Y \to X\) and \(G : Y \times X \to Y\) be two functions having the mixed monotone property. Assume that there exist non negative \(k, l\) with \(k + l < 1\) such that

\[d_X(F(x, y), F(u, v)) \leq kd_X(x, F(x, y)) + ld_Y(u, F(u, v)), \quad \forall x \geq_{P_1} u, y \leq_{P_2} v, \quad (2.13)\]

\[d_Y(G(y, x), G(v, u)) \leq kd_Y(y, G(y, x)) + ld_Y(v, G(v, u)), \quad \forall x \leq_{P_1} u, y \geq_{P_2} v. \quad (2.14)\]

If there exist \((x_0, y_0) \in X \times Y\) such that \(x_0 \leq_{P_1} F(x_0, y_0)\) and \(y_0 \geq_{P_2} G(y_0, x_0)\), then there exist \((x, y) \in X \times Y\) such that \(x = F(x, y)\) and \(y = G(y, x)\).

*Proof.* Following as in the proof of Theorem 2.7 it remains to show that \((x, y)\) is an FG-coupled fixed point. Recall from the proof of Theorem 2.7 that \(\{x_n\}\) is increasing in \(X\) and \(\{y_n\}\) is decreasing in \(Y\), \(\lim_{n \to \infty} F^n(x_0, y_0) = x\) and \(\lim_{n \to \infty} G^n(y_0, x_0) = y\).

We have

\[d_X(F(x, y), x) \leq d_X(F(x, y), F^n(x_0, y_0)) + d_X(F^n(x_0, y_0), x) = d_X(F^n(x_0, y_0), G^n(y_0, x_0)) + d_X(F^n(x_0, y_0), x).\]

By (i) and (ii) we have \(x \geq_{P_1} F^n(x_0, y_0)\) and \(y \leq_{P_2} G^n(y_0, x_0)\). Therefore using (2.13) we get

\[d_X(F(x, y), x) \leq kd_X(x, F(x, y)) + ld_X(F^n(x_0, y_0), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x).\]

As \(n \to \infty\),
\[d_X(F(x, y), x) \leq kd_X(x, F(x, y)).\]

This is possible if \(d_X(F(x, y), x) = 0\). Hence we have \(F(x, y) = x\).

Similarly using (2.14) we get \(d_Y(y, G(y, x)) \leq ld_Y(y, G(y, x)).\)

This is possible if \(d_Y(y, G(y, x)) = 0\). Hence \(G(y, x) = y\).

This completes the proof. \(\square\)

**Theorem 2.9.** Let \((X, d_X, \leq_{P_1})\) and \((Y, d_Y, \leq_{P_2})\) be two partially ordered complete metric spaces and \(F : X \times Y \to X\) and \(G : Y \times X \to Y\) be two continuous functions having the mixed monotone property. Assume that there exist \(k, l \in [0, \frac{1}{2})\) such that

\[d_X(F(x, y), F(u, v)) \leq kd_X(x, F(u, v)) + ld_X(u, F(x, y)), \quad \forall x \geq_{P_1} u, y \leq_{P_2} v, \quad (2.17)\]

\[d_Y(G(y, x), G(v, u)) \leq kd_Y(y, G(v, u)) + ld_Y(v, G(y, x)), \quad \forall x \leq_{P_1} u, y \geq_{P_2} v. \quad (2.18)\]
If there exist \((x_0, y_0) \in X \times Y\) such that \(x_0 \leq p_1 F(x_0, y_0)\) and \(y_0 \geq p_2 G(y_0, x_0)\), then there exist \((x, y) \in X \times Y\) such that \(x = F(x, y)\) and \(y = G(y, x)\).

**Proof.** As in Theorem 2.1 we can construct an increasing sequence \(\{x_n\}\) in \(X\) and a decreasing sequence \(\{y_n\}\) in \(Y\) where \(x_{n+1} = F(x_n, y_n) = F^{n+1}(x_0, y_0)\) and \(y_{n+1} = G(y_n, x_n) = G^{n+1}(y_0, x_0)\).

Claim: For \(n \in \mathbb{N}\)
\[
\begin{align*}
  d_X(F^{n+1}(x_0, y_0), F^n(x_0, y_0)) &\leq \left(\frac{l}{1-l}\right)^n d_X(x_0, x_0), \quad (2.19) \\
  d_Y(G^{n+1}(y_0, x_0), G^n(y_0, x_0)) &\leq \left(\frac{k}{1-k}\right)^n d_Y(y_1, y_0). \quad (2.20)
\end{align*}
\]

Using (2.17), (2.18) and symmetric property of \(d_Y\) we prove the claim.
For \(n = 1\), consider,
\[
\begin{align*}
  d_X(F^2(x_0, y_0), F(x_0, y_0)) &= d_X(F(F(x_0, y_0), G(y_0, x_0)), F(x_0, y_0)) \\
  &\leq kd_X(F(x_0, y_0), F(x_0, y_0)) + ld_X(x_0, F^2(x_0, y_0)) \\
  &\leq l[d_X(x_0, F(x_0, y_0)) + d_X(F(x_0, y_0), F^2(x_0, y_0))] \\
\end{align*}
\]
i.e., \((1-l)d_X(F^2(x_0, y_0), F(x_0, y_0)) \leq ld_X(x_0, F(x_0, y_0)) = ld_X(x_0, x_1)\)

i.e., \(d_X(F^2(x_0, y_0), F(x_0, y_0)) \leq \left(\frac{l}{1-l}\right)d_X(x_0, x_1)\).

For \(n = 1\), the claim is true.

Now assume the claim for \(n \leq m\) and check for \(n = m + 1\). Consider,
\[
\begin{align*}
  d_X(F^{m+2}(x_0, y_0), F^{m+1}(x_0, y_0)) &= d_X(F(F^{m+1}(x_0, y_0), G^{m+1}(y_0, x_0)), F(F^m(x_0, y_0), G^m(y_0, x_0))) \\
  &\leq kd_X(F^{m+1}(x_0, y_0), F^{m+1}(x_0, y_0)) + ld_X(F^m(x_0, y_0), F^{m+2}(x_0, y_0)) \\
  &\leq l[d_X(F^m(x_0, y_0), F^{m+1}(x_0, y_0)) + d_X(F^{m+1}(x_0, y_0), F^{m+2}(x_0, y_0))] \\
\end{align*}
\]
i.e., \((1-l)d_X(F^{m+2}(x_0, y_0), F^{m+1}(x_0, y_0)) \leq ld_X(F^m(x_0, y_0), F^{m+1}(x_0, y_0)) \leq l\left(\frac{l}{1-l}\right)^m d_X(x_0, x_1)\)

i.e., \(d_X(F^{m+2}(x_0, y_0), F^{m+1}(x_0, y_0)) \leq \left(\frac{l}{1-l}\right)^{m+1} d_X(x_0, x_1)\).

Similarly we get \(d_Y(G^{m+2}(y_0, x_0), G^{m+1}(y_0, x_0)) \leq \left(\frac{k}{1-k}\right)^{m+1} d_Y(y_0, y_1)\).

Thus the claim is true for all \(n \in \mathbb{N}\). Now using (2.19) and (2.20) we prove that \(\{F^n(x_0, y_0)\}\) and \(\{G^n(y_0, x_0)\}\) are Cauchy sequences in \(X\) and \(Y\) respectively.

For \(m > n\), consider,
\[
\begin{align*}
  d_X(F^m(x_0, y_0), F^n(x_0, y_0)) &\leq d_X(F^m(x_0, y_0), F^{m-1}(x_0, y_0)) + d_X(F^{m-1}(x_0, y_0), F^{m-2}(x_0, y_0)) + \cdots + d_X(F^{n+1}(x_0, y_0), F^n(x_0, y_0)) \\
\end{align*}
\]
and G we can prove that

\[ \text{Example 2.4.} \]

Let \( X \) and \( Y \) be two partially ordered complete metric spaces. Assume that \( X \) and \( Y \) have the following properties:

(i) if a non-decreasing sequence \( \{x_n\} \to x \) in \( X \), then \( x_n \leq x \) for all \( n \)

(ii) if a non-increasing sequence \( \{y_n\} \to y \) in \( Y \), then \( y_n \geq y \) for all \( n \).

Let \( F : X \times Y \to X \) and \( G : Y \times X \to Y \) be two functions having the mixed monotone property. Assume that there exist \( k \), \( l \in [0, 1) \) with

\[
\begin{align*}
  d_X(F(x,y), F(u,v)) &\leq kd_X(x, F(u,v)) + ld_X(u, F(x,y)), \quad \forall x \geq P_1 u, y \leq P_2 v; \quad (2.17) \\
  d_Y(G(y,x), G(v,u)) &\leq kd_Y(y, G(v,u)) + ld_Y(v, G(y,x)), \quad \forall x \leq P_1 u, y \geq P_2 v. \quad (2.18)
\end{align*}
\]

If there exist \( (x_0, y_0) \in X \times Y \) such that \( x_0 \leq P_1 F(x_0, y_0) \) and \( y_0 \geq P_2 G(y_0, x_0) \), then there exist \( (x,y) \in X \times Y \) such that \( x = F(x,y) \) and \( y = G(y,x) \).

**Proof.** Following as in the proof of Theorem 2.9 we only have to show that \( (x,y) \) is an FG-coupled fixed point. Recall from the proof of Theorem 2.9 that \( \{x_n\} \) is increasing in \( X \) and \( \{y_n\} \) is decreasing in \( Y \), \( \lim_{n \to \infty} F^n(x_0, y_0) = x \) and \( \lim_{n \to \infty} G^n(y_0, x_0) = y \). Now consider

\[
\begin{align*}
  d_X(F(x,y), x) &\leq d_X(F(x,y), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \\
  &= d_X(F(x,y), F(F^n(x_0, y_0), G^n(y_0, x_0))) + d_X(F^{n+1}(x_0, y_0), x).
\end{align*}
\]

By (i) and (ii) we have \( x \geq P_1 F^n(x_0, y_0) \) and \( y \leq P_2 G^n(y_0, x_0) \). Therefore using (2.17) we get

\[
\begin{align*}
  d_X(F(x,y), x) &\leq kd_X(x, F^{n+1}(x,y)) + ld_X(F^n(x_0, y_0), F(x,y)) + d_X(F^{n+1}(x_0, y_0), x).
\end{align*}
\]
As \( n \to \infty, d_X(F(x,y), x) \leq ld_X(x, F(x,y)) \).

This is possible if \( d_X(F(x,y), x) = 0 \). Hence we have \( F(x,y) = x \).

Similarly using (2.18) we prove that \( d_Y(y, G(y,x)) = 0 \). Thus \( G(y,x) = y \).

This completes the proof. \( \square \)

**Remark 2.6.** In all the theorems in this paper if we put \( X = Y \) and \( F = G \) we get several coupled fixed point theorems in partially ordered complete metric spaces.

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**REFERENCES**


ABSTRACT. We show that under certain conditions submatrices of row orbit matrices of strongly regular graphs span self-orthogonal codes. In order to demonstrate this method of construction, we construct self-orthogonal ternary linear codes from orbit matrices of the strongly regular graphs with parameters $(70,27,12,9)$. Also we construct non self-orthogonal binary linear codes from these orbit matrices. Further, we obtain strongly regular graphs and block designs from codewords of the constructed codes.

1. INTRODUCTION

We present a method for constructing self-orthogonal codes from submatrices of row orbit matrices of strongly regular graphs. Applying this method we construct self-orthogonal ternary linear codes from orbit matrices of strongly regular graph (SRG) with parameters $(70,27,12,9)$ for group $\mathbb{Z}_9$. We also construct non self-orthogonal binary linear codes from these matrices. We use the constructed codes to obtain strongly regular graphs and block designs. More precisely, the strongly regular graphs and block designs are constructed from codewords of a given weight of the obtained binary linear codes.

The paper is organized as follows: after a brief description of the terminology and some background results in Section 2, in Section 3 we describe the concept of orbit matrices of strongly regular graphs, based on results presented in [3, 8], and in Section 4 we present obtained orbit matrices of SRG(70,27,12,9) for group $\mathbb{Z}_9$. In Section 5 we present a method for construction of self-orthogonal codes from row orbit matrices of strongly regular graphs, and in Section 6 we construct binary and ternary codes from obtained orbit matrices. In Section 7 we construct strongly regular graphs and designs from codewords of the obtained codes.

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2. BACKGROUND AND TERMINOLOGY

We assume that the reader is familiar with basic notions from theory of finite groups. For basic definitions and properties of strongly regular graphs we refer the reader to [4] or [17].

A graph is regular if all its vertices have the same valency; a simple regular graph $\Gamma = (\mathcal{V}, \mathcal{E})$ is strongly regular with parameters $(v,k,\lambda,\mu)$ if it has $|\mathcal{V}| = v$ vertices, valency $k$, and if any two adjacent vertices are together adjacent to $\lambda$ vertices, while any two nonadjacent vertices are together adjacent to $\mu$ vertices. A strongly regular graph with parameters $(v,k,\lambda,\mu)$ is usually denoted by $\text{SRG}(v,k,\lambda,\mu)$. An automorphism of a strongly regular graph $\Gamma$ is a permutation of vertices of $\Gamma$, such that every two vertices are adjacent if and only if their images are adjacent.

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $I \subseteq \mathcal{P} \times \mathcal{B}$, is a $t$-$(v,b,r,k,\lambda)$ design, if $|\mathcal{P}| = v$, $|\mathcal{B}| = b$, every block $B \in \mathcal{B}$ is incident with precisely $k$ points, every $t$ distinct points are together incident with precisely $\lambda$ blocks and every point is incident with exactly $r$ blocks.

A linear $q$-ary $(n,k)$ code $K$ over the finite field $F_q$ of prime-power order $q$ is a $k$-dimensional subspace of the $n$-dimensional vector space over $F_q$. The weight of a codeword is the number of its elements that are nonzero and the distance between two codewords is the Hamming distance between them, that is, the number of elements in which they differ. The minimum distance between distinct codewords is denoted by $d$. The minimum distance of a linear code is the minimum weight of its nonzero codewords. If a linear code $K$ over a field of order $q$ is of length $n$, dimension $k$, and minimum distance $d = d(K)$, then we write $[n,k,d]_q$ to show this information. An $[n,k]$ linear code $K$ is said to be a best known linear $[n,k]$ code if $K$ has the highest minimum weight among all known $[n,k]$ linear codes. An $[n,k]$ linear code $K$ is said to be an optimal linear $[n,k]$ code if the minimum weight of $K$ achieves the theoretical upper bound on the minimum weight of $[n,k]$ linear codes, and near-optimal if its minimum distance is at most 1 less than the largest possible value.

The dual code $K^\perp$ is the orthogonal complement under the standard inner product $(\cdot,\cdot)$, i.e. $K^\perp = \{v \in F^n | (v, c) = 0 \text{ for all } c \in K\}$. If $K \subseteq K^\perp$, then $K$ is called self-orthogonal.

The support of a nonzero codeword $x = \{x_1, \ldots, x_n\}$ is the set of indices of its nonzero coordinates, i.e. $\text{supp}(x) = \{i \mid x_i \neq 0\}$. The support design of a code of length $n$ for a given nonzero weight $w$ is the design with points the $n$ coordinate indices and blocks the supports of all codewords of weight $w$.

3. ORBIT MATRICES OF STRONGLY REGULAR GRAPHS

In 2011 Behbahani and Lam introduced the concept of orbit matrices of SRGs (see [3]). While Behbahani and Lam were mostly focused on orbit matrices of
strongly regular graphs admitting an automorphism of prime order, a general definition of an orbit matrix of a strongly regular graph is given in [8].

Let $\Gamma$ be a SRG($v,k,\lambda,\mu$) and $A$ be its adjacency matrix. Suppose an automorphism group $G$ of $\Gamma$ partitions the set of vertices $V$ into $b$ orbits $O_1, \ldots, O_b$, with lengths $n_1, \ldots, n_b$, respectively. The orbits divide $A$ into submatrices $[A_{ij}]$, where $A_{ij}$ is the adjacency matrix of vertices in $O_i$ versus those in $O_j$. We define matrices $C = [c_{ij}]$ and $R = [r_{ij}]$, $1 \leq i, j \leq b$, such that

$c_{ij} = \text{column sum of } A_{ij},$
$r_{ij} = \text{row sum of } A_{ij}.$

The matrix $R$ is related to $C$ by

$$r_{ij}n_i = c_{ij}n_j. \quad (3.1)$$

Since the adjacency matrix is symmetric, it follows that

$$R = C^T. \quad (3.2)$$

The matrix $R$ is the row orbit matrix of the graph $\Gamma$ with respect to $G$, and the matrix $C$ is the column orbit matrix of the graph $\Gamma$ with respect to $G$. The matrices $C = [c_{ij}]$ and $R = [r_{ij}]$ satisfy the following conditions (see [8]):

$$\sum_{s=1}^{b} \frac{n_{s}}{n_{j}} c_{is} c_{js} = \delta_{ij}(k - \mu) + \mu n_{i} + (\lambda - \mu)c_{ij}$$

$$\sum_{s=1}^{b} \frac{n_{s}}{n_{j}} r_{si} r_{sj} = \delta_{ij}(k - \mu) + \mu n_{i} + (\lambda - \mu)r_{ji}.$$

Let us assume that a group $G$ acts as an automorphism group of a SRG($v,k,\lambda,\mu$). Each matrix with the properties of a matrix $R$ or $C$ will be called a row orbit matrix or a column orbit matrix, respectively, for a strongly regular graph with parameters ($v,k,\lambda,\mu$) and a group $G$ (see [3]) although not every orbit matrix gives rise to strongly regular graphs. The following definition of orbit matrices of strongly regular graphs was introduced in [8].

**Definition 3.1.** A $(b \times b)$-matrix $R = [r_{ij}]$ with entries satisfying conditions:

$$\sum_{j=1}^{b} r_{ij} = \sum_{i=1}^{b} \frac{n_{i}}{n_{j}} r_{ij} = k$$

$$\sum_{s=1}^{b} \frac{n_{s}}{n_{j}} r_{si} r_{sj} = \delta_{ij}(k - \mu) + \mu n_{i} + (\lambda - \mu)r_{ji}$$

where $0 \leq r_{ij} \leq n_{j}$, $0 \leq r_{ii} \leq n_{i} - 1$ and $\sum_{i=1}^{b} n_{i} = v$, is called a row orbit matrix for a strongly regular graph with parameters ($v,k,\lambda,\mu$) and the orbit lengths distribution $(n_1, \ldots, n_b)$. 
Definition 3.2. A \((b \times b)\)-matrix \(C = [c_{ij}]\) with entries satisfying conditions:
\[
\sum_{i=1}^{b} c_{ij} = \sum_{j=1}^{b} n_{ij} c_{ij} = k
\]
\[
\sum_{s=1}^{b} n_{js} c_{is} c_{js} = \delta_{ij}(k - \mu) + \mu n_{i} + (\lambda - \mu) c_{ij}
\]
where \(0 \leq c_{ij} \leq n_{i}\), \(0 \leq c_{ii} \leq n_{i} - 1\) and \(\sum_{i=1}^{b} n_{i} = v\), is called a column orbit matrix for a strongly regular graph with parameters \((v, k, \lambda, \mu)\) and the orbit lengths distribution \((n_{1}, \ldots, n_{b})\).

3.1. Orbit lengths distribution

Suppose an automorphism group \(G\) of \(\Gamma\) partitions the set of vertices \(V\) into \(b\) orbits \(O_{1}, \ldots, O_{b}\), with sizes \(n_{1}, \ldots, n_{b}\). Obviously, \(n_{i}\) is a divisor of \(|G|\), \(i = 1, \ldots, b\), and
\[
\sum_{i=1}^{b} n_{i} = v.
\]

When determining the orbit lengths distribution we also use the following result that can be found in [2].

**Theorem 3.1.** Let \(s < r < k\) be the eigenvalues of a SRG\((v, k, \lambda, \mu)\), then
\[
\phi \leq \max(\lambda, \mu) v,
\]
where \(\phi\) is the number of fixed points for an nontrivial automorphism.

3.2. Prototypes for a row of a column orbit matrix

To construct orbit matrices with parameters \((v, k, \lambda, \mu)\) and the orbit lengths distribution \((n_{1}, \ldots, n_{b})\) we first need to find all prototypes.

A prototype for a row of a column orbit matrix \(C\) gives the information about the number of occurrences of each integer as an entry of a particular row of \(C\). Behbahani and Lam [2, 3] introduced the concept of a prototype for a row of a column orbit matrix \(C\) of a strongly regular graph with a presumed automorphism group of prime order. We will generalize this concept, and describe a prototype for a row of a column orbit matrix \(C\) of a strongly regular graph under a presumed automorphism group of composite order.

Suppose an automorphism group \(G\) of a strongly regular graph \(\Gamma\) partitions the set of vertices \(V\) into \(b\) orbits \(O_{1}, \ldots, O_{b}\), of sizes \(n_{1}, \ldots, n_{b}\). With \(l_{i}, i = 1, \ldots, \rho\), we denote all divisors of \(|G|\) in ascending order \((l_{1} = 1, \ldots, l_{\rho} = |G|)\).

3.2.1. Prototypes for a fixed row

Consider the \(r\)-th row of a column orbit matrix \(C\). We say that it is a fixed row of a matrix \(C\) if \(n_{r} = 1\), i.e. if it corresponds to an orbit of length 1. The entries in
this row are either 0 or 1. Let $d_{l_i}$ denote the number of orbits whose length are $l_i$, $i = 1, \ldots, \rho$.

Let $x_e$ denote the number of occurrences of an element $e \in \{0, 1\}$ at the positions of the $r$-th row which correspond to the orbits of length 1. It follows that

$$x_0 + x_1 = d_1,$$

where $d_1$ is the number of orbits of length 1. Since the diagonal elements of the adjacency matrix of a strongly regular graphs are equal to 0, it follows that $x_0 \geq 1$.

Let $y_e^{(l_i)}$ denote the number of occurrences of an element $e \in \{0, 1\}$ at the positions of the $r$-th row which correspond to the orbits of length $l_i (i = 2, \ldots, \rho)$. We have

$$y_0^{(l_i)} + y_1^{(l_i)} = d_{l_i}, \quad i = 2, \ldots, \rho$$

Because the row sum of an adjacency matrix is equal to $k$, it follows that

$$x_1 + \sum_{i=2}^{\rho} l_i \cdot y_1^{(l_i)} = k.$$

The vector

$$p_1 = (x_0, x_1, y_0^{(l_2)}, y_1^{(l_2)}, \ldots, y_0^{(l_\rho)}, y_1^{(l_\rho)})$$

whose components are nonnegative integer solutions of the equalities (3.3), (3.4) and (3.5) is called a prototype for a fixed row. The length of a prototype for a fixed row is $2\rho$.

### 3.2.2. Prototypes for a nonfixed row

Let us consider the $r$-th row of a column orbit matrix $C$, where $n_r \neq 1$. Let $d_{l_i}$ denote the number of orbits whose length is $l_i$, $i = 1, \ldots, \rho$.

If a fixed vertex is adjacent to a vertex from an orbit $O_i, 1 \leq i \leq b$, then it is adjacent to all vertices from the orbit $O_i$. Therefore, the entries at the positions corresponding to fixed columns are either 0 or $n_r$. Let $x_e$ denote the number of occurrences of an element $e \in \{0, n_r\}$ at those positions of the $r$-th row which correspond to the orbits of length 1. We have

$$x_0 + x_{n_r} = d_1.$$

The entries at the positions corresponding to the orbits whose lengths are greater than 1 are 0, 1, \ldots, $n_r - 1$ or $n_r$. The entry at the position $(r, r)$ is $0 \leq c_{rr} \leq n_r - 1$, since the diagonal elements of the adjacency matrix of strongly regular graphs are 0.

Let $y_e^{(l_i)}$ denote the number of occurrences of an element $e \in \{0, \ldots, n_r\}$ of $r$-th row at the positions which correspond to the orbits of length $l_i (i = 2, \ldots, \rho)$. From (3.1) and (3.2) we conclude that

$$c_{rr} n_i = c_{ir} n_r,$$

where $c_{ir} \in \{0, \ldots, n_r\}$. If $c_{ri} \cdot \frac{n_r}{n_r} \not\in \{0, \ldots, n_i\}$, then $y_{e^{(n_i)}} = 0$. It follows that

$$\sum_{e=0}^{n_r} y_e^{(l_i)} = d_{l_i}, \quad i = 2, \ldots, \rho.$$
Since the row sum of an adjacency matrix is equal to \( k \), we have that
\[
x_{n_r} + \sum_{i=2}^{\rho} \sum_{h=1}^{n_r} y_i^{(h)} \cdot h \cdot \frac{n_i}{n_r} = k, \tag{3.8}
\]
If \( s_{ij} = \sum_{k=1}^{b} c_{ik} c_{jk} n_k \), then \( s_{rr} = \sum_{k=1}^{b} c_{rk} c_{rk} n_k \), and from the definition 3.2 we have that
\[
n_r^2 x_{n_r} + \sum_{i=2}^{\rho} \sum_{h=1}^{n_r} y_i^{(h)} \cdot h^2 \cdot n_l_i = s_{rr}, \tag{3.9}
\]
where \( s_{rr} = (k - \mu)n_r + \mu n_r^2 + (\lambda - \mu)c_{rr} n_r \) and \( c_{rr} \in \{0, \ldots, n_r - 1\} \).

The vector
\[
p_{n_r} = (x_0, x_{n_r}, y_0^1, \ldots, y_{n_r}^1; \ldots; y_0^\rho, \ldots, y_{n_r}^\rho),
\]
whose components are nonnegative integer solutions of equalities (3.6), (3.7), (3.8) and (3.9) is called a prototype for a row corresponding to the orbit of length \( n_r \).

The length of a prototype for a row which corresponds to the orbit of length \( n_r \) is
\[
2 + \sum_{i=2}^{\rho} (n_r + 1).
\]

4. Orbit Matrices of SRG(70,27,12,9)

Let \( \Gamma \) be a strongly regular graph with parameters (70,27,12,9). Further, let us assume that the group \( G \cong \langle a \mid a^9 = 1 \rangle \cong Z_9 \) acts as an automorphism group of \( \Gamma \). By \( d_i \) we denote the number of \( G \)-orbits of length \( i, i \in \{1,3,9\} \), and by \( d = (d_1, d_3, d_9) \) we denote the corresponding orbit lengths distribution. To find all orbit matrices of SRG(70,27,12,9) for group \( Z_9 \) first we find all the orbit lengths distributions \((n_1, n_2, \ldots, n_b)\) for an action of the group \( Z_9 \) that satisfy Theorem 3.1. Using the program Mathematica we get all the prototypes for every orbit length distribution.

Using our own programs written in GAP [11] we construct all orbit matrices for given orbit lengths distributions. In Table 1 we present the number of orbit matrices for \( Z_9 \) for each orbit lengths distribution.

<table>
<thead>
<tr>
<th>distribution # OM</th>
<th>distribution # OM</th>
<th>distribution # OM</th>
<th>distribution # OM</th>
</tr>
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<td>1</td>
</tr>
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<td>(7, 3, 6)</td>
<td>1</td>
</tr>
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<td>(1, 8, 5)</td>
<td>0</td>
<td>(7, 6, 5)</td>
<td>1</td>
</tr>
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<td>(7, 9, 4)</td>
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</tr>
<tr>
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</tr>
<tr>
<td>(4, 10, 4)</td>
<td>7</td>
<td>(13, 1, 6)</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1. Number of nonisomorphic orbit matrices of SRGs with parameters (70,27,12,9) for an automorphism group \( Z_9 \).
5. SELF ORTHOGONAL CODES FROM ORBIT MATRICES OF STRONGLY REGULAR GRAPHS

In 2003 Harada and Tonchev introduced a method of constructing self-orthogonal codes from orbit matrices of a design (see [14]). In [8] a method for constructing self-orthogonal codes from column orbit matrices of strongly regular graphs admitting an automorphism group \( G \) which acts with all orbits of the same length is described. These codes were defined over \( F_q \), a finite field of prime order \( q \), such that \( q \) divides \( k, \lambda, \mu \). Methods of constructing self-orthogonal codes from row orbit matrices of strongly regular graphs are given in [9]. Here we give a construction of self-orthogonal codes from some submatrices of row orbit matrices of strongly regular graphs.

**Theorem 5.1.** Let \( \Gamma \) be a \( \text{SRG}(v, k, \lambda, \mu) \) having an automorphism group \( G \) which acts on the set of vertices of \( \Gamma \) with \( b \) orbits of lengths \( n_1, \ldots, n_b \), respectively, such that \( n_1 = n_2 = \ldots = n_f = 1 \) and \( p \mid n_s \) if \( n_s > 1 \). Further, let \( p \) be a prime dividing \( k, \lambda, \mu \). Let \( R \) be the row orbit matrix of the graph \( \Gamma \) with respect to \( G \). If \( p \) is a prime then the code over \( F_p \) spanned by the fixed rows of \( R \) is a self-orthogonal code of length \( b \).

**Proof.** From the definition of an orbit matrix, for \( i^{th} \) and \( j^{th} \) rows of \( C \) we have

\[
\sum_{s=1}^{b} \frac{n_s}{n_j} c_{js} c_{is} = \delta_{ij}(k - \mu) + \mu n_i + (\lambda - \mu) c_{ij}.
\]

(5.1)

Let \( n_i = n_j = 1 \). We have

\[
\sum_{s=1}^{b} \frac{n_s}{n_j} c_{js} c_{is} = \sum_{s, n_i=1}^{b} \frac{n_s}{n_j} c_{js} c_{is} + \sum_{s, n_i>1} n_s c_{js} c_{is}.
\]

so

\[
\sum_{s, n_i=1}^{b} c_{js} c_{is} = \sum_{s=1}^{b} \frac{n_s}{n_j} c_{js} c_{is} - \sum_{s, n_i>1} n_s c_{js} c_{is} = \delta_{ij}(k - \mu) + \mu n_i + (\lambda - \mu) c_{ij} - \sum_{s, n_i>1} n_s c_{js} c_{is}.
\]

Because \( p \mid n_s \) we conclude that \( \sum_{s, n_i=1}^{b} c_{js} c_{is} \) is congruent to zero modulo \( p \).

If \( n_i = n_j = 1 \) then

\[
\sum_{s=1}^{b} r_{is} r_{js} = \sum_{s, n_i=1}^{b} r_{is} r_{js} + \sum_{s, n_i>1} r_{is} r_{js}.
\]

Since \( r_{is} = c_{is} \frac{n_s}{n_i} = c_{is} n_s \) we have

\[
\sum_{s=1}^{b} r_{is} r_{js} = \sum_{s, n_i=1}^{b} c_{is} c_{js} + \sum_{s, n_i>1} n_s c_{is} r_{js}.
\]

(5.2)

From (5.1) and (5.2) and because \( p \mid n_s \) we conclude that \( \sum_{s=1}^{b} r_{is} r_{js} \) is congruent to zero modulo \( p \). \( \square \)
Theorem 5.2. Let $\Gamma$ be a SRG$(v,k,\lambda,\mu)$ having an automorphism group $G$ which acts on the set of vertices of $\Gamma$ with $b$ orbits of lengths $n_1, \ldots, n_b$, respectively, such that there are $f$ fixed vertices, $h$ orbits of length $w$, and $b - f - h$ orbits of lengths $n_{f+w+h+1}, \ldots, n_b$. Further, let $pw|n$, if $w < n$, and $pn|w$ if $n < w$, for $s = f + h + 1, \ldots, b$, where $p$ is a prime number dividing $k, \lambda, \mu$ and $w$. Let $R$ be the row orbit matrix of the graph $\Gamma$ with respect to $G$. If $q$ is a prime power such that $q = p^m$, then the code over $F_q$ spanned by the part of the matrix $R$ (rows and columns) determined by the orbits of length $w$ is a self-orthogonal code of length $h$. If $m = \min\{w, n_{f+w+h+1}, \ldots, n_b\}$, such that $p|m$ and $pm|n_s$ if $n_s \neq m$, then the code over $F_q$ spanned by the rows of $R$ corresponding to the orbits of length $m$ and columns corresponding to the orbits of length greater than $m - 1$ is a self-orthogonal code of length $b - f$.

Proof. Let $n_i = n_j = w$. Then

$$\sum_{s=1}^{b} \frac{n_s}{n_j} c_{js} c_{is} = \sum_{s, n_s = 1}^{n_s} \frac{n_s}{n_j} c_{js} c_{is} + \sum_{s, 1 < n_s < w} \frac{n_s}{n_j} c_{js} c_{is} + \sum_{n_s = w}^{n_s} \frac{n_s}{n_j} c_{js} c_{is}$$

So

$$\sum_{s, n_s = w}^{n_s} c_{js} c_{is} = \sum_{s, 1 < n_s < w}^{n_s} \frac{n_s}{n_j} c_{js} c_{is} - \frac{1}{w} \sum_{s, 1 < n_s < w}^{n_s} c_{js} c_{is} - \sum_{s, n_s > w}^{n_s} \frac{n_s}{n_j} c_{js} c_{is}$$

If $s \in \{1, 2, \ldots, f\}$, then $c_{js} c_{is} \in \{0, w\}$, so $c_{js} c_{is} \in \{0, w^2\}$. From (5.1) and (5.3) it follows that:

$$\sum_{s, n_s = w}^{n_s} c_{js} c_{is} = \delta_{ij}(k - \mu) + \mu n_i + (\lambda - \mu) c_{ij} - w \cdot (x - \lambda) - \sum_{s, 1 < n_s < w}^{n_s} \frac{n_s}{w} c_{js} c_{is} - \sum_{s, n_s > w}^{n_s} \frac{n_s}{w} c_{js} c_{is},$$

where $x = |\{s \in \{1, \ldots, f\} : c_{js} c_{is} = 0\}|$. If $n_s > w$ then $p|n_s$. If $1 < n_s < w$ then $p|n_s$, because $c_{is} = r_{is} n_s = r_{is} w, c_{js} = r_{js} n_j = r_{js} w$, and $p|w$.

Hence $\sum_{s, n_s = w}^{n_s} c_{js} c_{is}$ is congruent to zero modulo $p$.

Since $r_{is} = c_{is} \frac{n_s}{w}$ and $r_{js} = c_{js} \frac{n_s}{w}$, we have that

$$\sum_{s, n_s = w}^{n_s} r_{js} r_{is} = \sum_{s, n_s = w}^{n_s} c_{js} \frac{n_s}{n_j} c_{is} \frac{n_s}{n_i} = \sum_{s, n_s = w}^{n_s} c_{js} c_{is}$$

so $\sum_{s, n_s = w}^{n_s} r_{js} r_{is}$ is congruent to zero modulo $p$. Let $n_i = n_j = m$. We have

$$\sum_{s = f + 1}^{b} r_{js} r_{is} = \sum_{s, n_s = m}^{n_s} r_{js} r_{is} + \sum_{s, n_s > m}^{n_s} r_{js} r_{is}.$$ 

Since $r_{is} = c_{is} \frac{n_s}{n_i}$ it follows that $r_{js} r_{is}$ is divisible by $p$ if $n_s > m$.

Also

$$\sum_{s, n_s = m}^{n_s} r_{js} r_{is} = \sum_{s, n_s = m}^{n_s} c_{js} c_{is} = \delta_{ij}(k - \mu) + \mu n_i + (\lambda - \mu)c_{ij} - m \cdot (f - y) - \sum_{s, n_s > m}^{n_s} \frac{n_s}{m} c_{js} c_{is},$$

where $y = |\{s \in \{1, \ldots, f\} : c_{js} c_{is} = 0\}|$.

Hence, we conclude that $\sum_{s, n_s = m}^{n_s} r_{js} r_{is}$ is divisible by $p$ and $\sum_{s = f + 1}^{b} r_{js} r_{is}$ is congruent to zero modulo $p$. \qed
6. Linear Codes from Orbit Matrices of SRG(70,27,12,9)

In this section we construct self-orthogonal codes from orbit matrices of SRG(70,27,12,9) for group $Z_9$ presented in the Section 4 by applying theorems presented in Section 5. First we construct ternary self-orthogonal codes from the orbit matrices. Also we construct non self-orthogonal binary codes from the orbit matrices. In Tables 2, 3, 4, 5, 6, 7 and 8 we present information for the obtained codes, omitting the trivial codes. The codes were analyzed using Magma [5]. Codes marked with $\ast$ are optimal linear codes.

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Table 2. Ternary codes from orbit matrices of SRG(70,24,12,9) for $Z_9$ obtained from part corresponding to the orbits of length 3

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<th>$\text{distribution}$</th>
<th>$[n,k,d]$</th>
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<td>(1,11,4)</td>
<td>[15,4,6]</td>
<td>576</td>
</tr>
<tr>
<td>(1,11,4)</td>
<td>[15,4,6]$\ast$</td>
<td>1728</td>
<td>(4,4,6)</td>
<td>[10,1,3]</td>
<td>30240</td>
</tr>
<tr>
<td>(4,7,5)</td>
<td>[12,3,3]</td>
<td>2160</td>
<td>(7,6,5)</td>
<td>[11,2,3]</td>
<td>8640</td>
</tr>
<tr>
<td>(7,9,4)</td>
<td>[14,3,3]</td>
<td>1728</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3. Ternary codes from orbit matrices of SRG(70,24,12,9) for $Z_9$ obtained from rows corresponding to the orbits of length 3 and columns corresponding to the orbits of length 3 and 9

<table>
<thead>
<tr>
<th>distribution</th>
<th>$[n,k,d]$</th>
<th>$\text{Aut}(K)$</th>
<th>$\text{distribution}$</th>
<th>$[n,k,d]$</th>
<th>$\text{Aut}(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,2,7)</td>
<td>[7,3,3]</td>
<td>6</td>
<td>(1,2,7)</td>
<td>[7,2,3]</td>
<td>12</td>
</tr>
<tr>
<td>(1,2,7)</td>
<td>[7,2,3]</td>
<td>18</td>
<td>(1,2,7)</td>
<td>[7,2,3]</td>
<td>24</td>
</tr>
<tr>
<td>(1,2,7)</td>
<td>[7,2,3]</td>
<td>72</td>
<td>(4,1,7)</td>
<td>[7,1,6]</td>
<td>72</td>
</tr>
<tr>
<td>(4,1,7)</td>
<td>[7,1,6]</td>
<td>720</td>
<td>(4,1,7)</td>
<td>[6,1,6]$\ast$</td>
<td>72</td>
</tr>
</tbody>
</table>

Table 4. Ternary codes from orbit matrices of SRG(70,24,12,9) for $Z_9$ obtained from part corresponding to the orbits of length 9

<table>
<thead>
<tr>
<th>distribution</th>
<th>$[n,k,d]$</th>
<th>$\text{Aut}(K)$</th>
<th>$\text{distribution}$</th>
<th>$[n,k,d]$</th>
<th>$\text{Aut}(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,11,4)</td>
<td>[11,8,1]</td>
<td>288</td>
<td>(1,11,4)</td>
<td>[11,8,1]</td>
<td>192</td>
</tr>
<tr>
<td>(1,11,4)</td>
<td>[11,8,1]</td>
<td>288</td>
<td>(1,11,4)</td>
<td>[11,8,1]</td>
<td>288</td>
</tr>
<tr>
<td>(1,11,4)</td>
<td>[11,8,1]$\ast$</td>
<td>576</td>
<td>(4,4,6)</td>
<td>[4,2,1]</td>
<td>6</td>
</tr>
<tr>
<td>(4,7,5)</td>
<td>[7,4,1]</td>
<td>24</td>
<td>(4,7,5)</td>
<td>[7,4,1]</td>
<td>144</td>
</tr>
<tr>
<td>(4,7,5)</td>
<td>[7,6,1]</td>
<td>240</td>
<td>(4,10,4)</td>
<td>[10,8,1]</td>
<td>4320</td>
</tr>
<tr>
<td>(4,10,4)</td>
<td>[10,8,1]</td>
<td>10080</td>
<td>(4,10,4)</td>
<td>[10,8,1]</td>
<td>10080</td>
</tr>
<tr>
<td>(4,10,4)</td>
<td>[10,8,1]</td>
<td>80640</td>
<td>(7,9,4)</td>
<td>[9,6,2]$\ast$</td>
<td>1296</td>
</tr>
<tr>
<td>(7,9,4)</td>
<td>[9,8,2]$\ast$</td>
<td>362880</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5. Binary codes from orbit matrices of SRG(70,24,12,9) for $Z_9$ obtained from part corresponding to the orbits of length 3
In this section we use the codes constructed in Section 6 to obtain strongly regular graphs and block designs. In order to construct strongly regular graphs we consider a set of codewords of certain weight \(w\) and look at the pairwise distances of the codewords. We identify the vertices of the graph by the codewords of weight \(w\) and define adjacency with respect to the Hamming distance of the codewords. In some cases the constructed graphs are strongly regular. We observe the cases when the distances between two codewords take two, three or four values.

If there are two possible values for a distance between two codewords, denoted by \(d_1\) and \(d_2\), then we define two vertices \(x\) and \(y\) to be adjacent if and only if
$d(x,y) = d_1$ (for the complementary graph we define that $x$ and $y$ are adjacent if and only if $d(x,y) = d_2$).

If there are three possible values for a distance between two codewords, namely $d_1$ and $d_2$ and $d_3$, we have more possibilities to define adjacency. Firstly, we define two vertices $x$ and $y$ to be adjacent if and only if $d(x,y) = d_1$, Secondly, two vertices $x$ and $y$ are adjacent if and only if $d(x,y) = d_2$, and thirdly, two vertices $x$ and $y$ are adjacent if and only if $d(x,y) = d_3$.

Let there be four values for a distance between two codewords, $d_1$ and $d_2$, $d_3$ and $d_4$. Firstly, we define two vertices $x$ and $y$ to be adjacent if and only if $d(x,y) = d_1$, or $d(x,y) = d_2$, secondly, two vertices $x$ and $y$ are adjacent if and only if $d(x,y) = d_1$ or $d(x,y) = d_3$ and thirdly, two vertices $x$ and $y$ are adjacent if and only if $d(x,y) = d_1$ or $d(x,y) = d_4$. Further, we define adjacency taking into consideration only one intersection ($d_1$, $d_2$, $d_3$ or $d_4$). The construction is conducted using the GAP package Grape [16]. The obtained strongly regular graphs are presented in Tables 9, 10, 11.

### Table 9. SRGs from binary codes obtained from part corresponding to the orbits of length 3

| $(v,k,\lambda,\mu)$ | $|\text{Aut}(G)|$ | Distribution |
|---------------------|-------------------|--------------|
| (15,6,1,3)         | 720               | (7,6,5)      |
| (21,10,3,6)        | 5040              | (4,10,4)     |
| (27,10,1,5)        | 51840             | (7,9,4)      |
| (28,12,6,4)        | 40320             | (4,10,4)     |
| (36,14,7,4)        | 362880            | (7,9,4)      |
| (45,16,8,4)        | 3628800           | (4,10,4)     |
| (120,56,28,24)     | 3628800           | (7,9,4)      |
| (126,100,78,84)    | 3628800           | (7,9,4)      |

### Table 10. SRGs from binary codes obtained from rows corresponding to the orbits of length 3 and columns corresponding to the orbits of length 3 and 9

| $(v,k,\lambda,\mu)$ | $|\text{Aut}(G)|$ | Distribution |
|---------------------|-------------------|--------------|
| (10,3,0,1)          | 120               | (1,5,6)      |
| (15,6,1,3)          | 720               | (1,2,7)      |
| (28,12,6,4)         | 40320             | (1,5,6),(4,1,7) |
| (36,14,7,4)         | 362880            | (7,3,6)      |
| (45,16,8,4)         | 3628800           | (4,4,6)      |
| (55,18,9,4)         | 39916800          | (7,6,5)      |
| (78,22,11,4)        | 6227020800        | (7,9,4)      |
| (119,54,21,27)      | 394813440         | (4,7,5)      |
| (120,56,28,24)      | 3628800           | (4,4,6)      |
| (126,100,78,84)     | 3628800           | (7,3,6)      |
| (330,266,211,228)   | 39916800          | (7,6,5)      |

### Table 11. SRGs from binary codes obtained from part corresponding to the orbits of length 9

| $(v,k,\lambda,\mu)$ | $|\text{Aut}(G)|$ | Distribution |
|---------------------|-------------------|--------------|
| (10,3,0,1)          | 120               | (4,7,5),(7,6,5) |
| (15,6,1,3)          | 720               | (1,5,6),(4,4,6),(7,3,6) |
| (21,10,3,6)         | 5040              | (1,2,7),(4,1,7),(7,0,7) |
| (35,16,6,8)         | 40320             | (1,2,7),(4,1,7),(7,0,7) |
The strongly regular graph with parameters \((10, 3, 0, 1)\) is the Petersen graph, the unique SRG with these parameters. The constructed strongly regular graphs with parameters \((15, 6, 1, 3)\) and \((21, 10, 3, 6)\) are complement of the triangular graphs \(T(6)\) and \(T(7)\). Strongly regular graphs with parameters \((28, 12, 6, 4)\), \((36, 14, 7, 4)\), \((45, 16, 8, 4)\), \((55, 18, 9, 4)\) and \((78, 22, 11, 4)\) are the triangular graphs \(T(8)\), \(T(9)\), \(T(10)\), \(T(11)\) and \(T(13)\) respectively. The constructed strongly regular graph with parameters \((35, 16, 6, 8)\) is the complement of the distance 2 graph in the Johnson graph \(J(7, 4)\) and \((27, 10, 1, 5)\) is a complement of a Schläfli graph. The constructed strongly regular graph with parameters \((120, 56, 28, 24)\) is the complement of the distance 2 graph in the Johnson graph \(J(10, 3)\) and strongly regular graph with parameters \((126, 100, 78, 84)\) is the complement of the distance 1 or 4 in the Johnson graph \(J(9, 4)\). The constructed strongly regular graph with parameters \((330, 266, 211, 228)\) is the complement of the distance 1 or 4 in the Johnson graph \(J(11, 4)\) and the strongly regular graph with parameters \((119, 54, 21, 27)\) is \(O(8, 2)\) polar graph.

We also use the codes from Section 6 to construct designs, taking into consideration a set of codewords of certain weight. Identifying coordinate positions of the codewords, and codewords of a weight \(k\) with blocks, we obtain an incidence structure having all blocks of size \(k\). In other words, we consider the support designs of the constructed codes. In some cases these support designs are designs. In Tables 12, 13 and 14 we present information on the obtained \(t - (v, b, r, k, \lambda)\) designs for which \(t < k < v - t\).

**Table 12. Designs from binary codes obtained from part corresponding to the orbits of length 3**

| \(t - (v, b, r, k, \lambda)\) | \(|\text{Aut}(D)|\) | Distribution | \(t - (v, b, r, k, \lambda)\) | \(|\text{Aut}(D)|\) | Distribution |
|---------------------------|----------------|--------------|---------------------------|----------------|--------------|
| 2-(10,120,36,3,8)         | 3628800        | (4,10,4)     | 2-(10,120,84,7,56)        | 3628800        | (4,10,4)     |
| 2-(10,210,126,6,70)       | 3628800        | (4,10,4)     | 2-(10,210,84,4,28)        | 3628800        | (4,10,4)     |
| 2-(10,252,126,5,56)       | 3628800        | (4,10,4)     | 3-(10,210,126,6,35)       | 3628800        | (4,10,4)     |
| 2-(10,210,84,4,47)        | 3628800        | (4,10,4)     | 3-(10,252,126,5,21)       | 3628800        | (4,10,4)     |
| 2-(6,20,10,3,4)           | 720            | (7,6,5)      | 2-(9,126,56,4,21)         | 362880         | (7,9,4)      |
| 2-(9,84,56,6,35)          | 362880         | (7,9,4)      | 3-(9,126,56,4,6)          | 362880         | (7,9,4)      |

**Table 13. Designs from binary codes obtained from rows corresponding to the orbits of length 3 and columns corresponding to the orbits of length 3 and 9**

| \(t - (v, b, r, k, \lambda)\) | \(|\text{Aut}(D)|\) | Distribution | \(t - (v, b, r, k, \lambda)\) | \(|\text{Aut}(D)|\) | Distribution |
|---------------------------|----------------|--------------|---------------------------|----------------|--------------|
| 2-(8,56,21,3,6)           | 40320          | (4,1,7)      | 2-(8,70,35,4,15)          | 40320          | (4,1,7)      |
| 3-(8,70,35,4,5)           | 40320          | (4,1,7)      | 2-(10,120,36,3,8)         | 3628800        | (4,4,6)     |
| 2-(10,210,84,4,28)        | 3628800        | (4,4,6)      | 2-(10,252,126,5,56)       | 3628800        | (4,4,6)     |
| 3-(10,210,126,6,35)       | 3628800        | (4,4,6)      | 3-(10,210,126,6,35)       | 3628800        | (4,4,6)     |
| 3-(10,252,126,5,21)       | 3628800        | (4,4,6)      | 3-(9,126,56,4,6)          | 362880         | (7,7,6)     |
| 3-(9,126,56,4,6)          | 362880         | (7,7,6)      | 2-(11,165,120,8,84)       | 3991680        | (7,6,5)     |
| 2-(11,330,120,4,36)       | 3991680        | (7,6,5)      | 2-(11,165,252,6,126)      | 3991680        | (7,6,5)     |
| 3-(11,330,120,4,8)        | 3991680        | (7,6,5)      | 3-(11,165,252,6,56)       | 3991680        | (7,6,5)     |
| 2-(13,128,792,8,462)      | 6227020800     | (7,9,4)      | 2-(13,1716,792,6,300)     | 6227020800     | (7,9,4)     |
| 2-(13,128,220,10,165)     | 6227020800     | (7,9,4)      | 2-(13,1716,220,4,55)      | 6227020800     | (7,9,4)     |
| 3-(13,1287,792,8,252)     | 6227020800     | (7,9,4)      | 3-(13,1716,792,6,120)     | 6227020800     | (7,9,4)     |
| 3-(13,715,220,4,10)       | 6227020800     | (7,9,4)      |                           |                |              |
| $t - \{v, b, r, k, \lambda\}$ | $|\text{Aut}(D)|$ | Distribution |
|-------------------------------|------------------|-------------|
| $2-(7,35,20,4,10)$            | 5040             | $(1,2,7),(4,1,7),(7,0,7)$ |
| $2-(6,20,10,3,4)$             | 720              | $(1,5,6),(4,4,6),(7,3,6)$ |

Table 14. Designs from binary codes obtained from part corresponding to the orbits of length 9

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REFERENCES
